

THE STEINBERG VARIETY AND REPRESENTATIONS OF REDUCTIVE GROUPS

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Dedicated to Gus Lehrer on the occasion of his 60th birthday

ABSTRACT. We give an overview of some of the main results in geometric representation theory that have been proved by means of the Steinberg variety. Steinberg's insight was to use such a variety of triples in order to prove a conjectured formula by Grothendieck. The Steinberg variety was later used to give an alternative approach to Springer's representations and played a central role in the proof of the Deligne-Langlands conjecture for Hecke algebras by Kazhdan and Lusztig.

1. INTRODUCTION

Suppose G is a connected, reductive algebraic group defined over an algebraically closed field k , \mathcal{B} is the variety of Borel subgroups of G , and u is a unipotent element in G . Let \mathcal{B}_u denote the closed subvariety of \mathcal{B} consisting of those Borel subgroups that contain u , let r denote the rank of G , and let C denote the conjugacy class of u .

In 1976, motivated by the problem of proving the equality conjectured by Grothendieck

$$(*) \quad \dim Z_G(u) = r + 2 \dim \mathcal{B}_u,$$

in order to get the multiplicity 2 in $(*)$ in the picture, Steinberg [Ste76] introduced a variety of triples

$$S = \{ (v, B, B') \in C \times \mathcal{B} \times \mathcal{B} \mid v \in B \cap B' \}.$$

By analyzing the geometry of the variety S , he was able to prove $(*)$ in most cases. In addition, by exploiting the fact that the G -orbits on $\mathcal{B} \times \mathcal{B}$ are canonically indexed by elements of the Weyl group of G , he showed that S could be used to establish relationships between Weyl group elements and unipotent elements in G .

Now let \mathfrak{g} denote the Lie algebra of G , and let \mathfrak{N} denote the variety of nilpotent elements in \mathfrak{g} . The *Steinberg variety* of G is

$$Z = \{ (x, B, B') \in \mathfrak{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \text{Lie}(B) \cap \text{Lie}(B') \}.$$

If the characteristic of k is zero or good for G , then there is a G -equivariant isomorphism between \mathfrak{N} and \mathcal{U} , the variety of unipotent elements in G , and so $Z \cong \{ (u, B, B') \in \mathcal{U} \times \mathcal{B} \times \mathcal{B} \mid u \in B \cap B' \}$.

In the thirty years since Steinberg first exploited the variety S , the Steinberg variety has played a key role in advancing our understanding of objects that at first seem to be quite unrelated:

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- Representations of the Weyl group W of G
- The geometry of nilpotent orbits in \mathfrak{g} and their covers
- Differential operators on \mathcal{B}
- Primitive ideals in the universal enveloping algebra of \mathfrak{g}
- Representations of p -adic groups and the local Langlands program

In this paper we hope to give readers who are familiar with some aspects of the representation theory of semisimple algebraic groups, or Lie groups, but who are not specialists in this particular flavor of geometric representation theory, an overview of the main results that have been proved using the Steinberg variety. In the process we hope to make these results more accessible to non-experts and at the same time emphasize the unifying role played by the Steinberg variety. We assume that the reader is quite familiar with the basics of the study of algebraic groups, especially reductive algebraic groups and their Lie algebras, as contained in the books by Springer [Spr98] and Carter [Car85] for example.

We will more or less follow the historical development, beginning with concrete, geometric constructions and then progressing to increasingly more advanced and abstract notions.

In §2 we analyze the geometry of Z , including applications to orbital varieties, characteristic varieties and primitive ideals, and generalizations.

In §3 we study the Borel-Moore homology of Z and the relation with representations of Weyl groups. Soon after Steinberg introduced his variety S , Kazhdan and Lusztig [KL80] defined an action of $W \times W$ on the top Borel-Moore homology group of Z . Following a suggestion of Springer, they showed that the representation of $W \times W$ on the top homology group, $H_{4n}(Z)$, is the two-sided regular representation of W . Somewhat later, Ginzburg [Gin86] and independently Kashiwara and Tanisaki [KT84], defined a multiplication on the total Borel-Moore homology of Z . With this multiplication, $H_{4n}(Z)$ is a subalgebra isomorphic to the group algebra of W .

The authors [DR08a] [DR08b] have used Ginzburg's construction to describe the top Borel-Moore homology groups of the generalized Steinberg varieties $X_{0,0}^{\mathcal{P},\mathcal{Q}}$ and $X_{\text{reg,reg}}^{\mathcal{P},\mathcal{Q}}$ (see §2.4) in terms of W , as well as to give an explicit, elementary, computation of the total Borel-Moore homology of Z as a graded algebra: it is isomorphic to the smash product of the coinvariant algebra of W and the group algebra of W .

Orbital varieties arise naturally in the geometry of the Steinberg variety. Using the convolution product formalism, Hinich and Joseph [HJ05] have recently proved an old conjecture of Joseph about inclusions of closures of orbital varieties.

In §4 we study the equivariant K -theory of Z and what is undoubtedly the most important result to date involving the Steinberg variety: the Kazhdan-Lusztig isomorphism [KL87] between $K^{G \times \mathbb{C}^*}(Z)$ and the extended, affine Hecke algebra \mathcal{H} . Using this isomorphism, Kazhdan and Lusztig were able to classify the irreducible representations of \mathcal{H} and hence to classify the representations containing a vector fixed by an Iwahori subgroup of the p -adic group with the same type as the Langlands dual ${}^L G$ of G . In this way, the Steinberg variety plays a key role in the local Langlands program and also leads to a better understanding of the extended affine Hecke algebra.

Very recent work involving the Steinberg variety centers around attempts to categorify the isomorphism between the specialization of $K^{G \times \mathbb{C}^*}(Z)$ at p and the Hecke algebra of Iwahori bi-invariant functions on ${}^L G(\mathbb{Q}_p)$. Because of time and space constraints, we leave a discussion of this research to a future article.

2. GEOMETRY

For the rest of this paper, in order to simplify the exposition, we assume that G is connected, the derived group of G is simply connected, and that $k = \mathbb{C}$. Most of the results below hold, with obvious modifications, for an arbitrary reductive algebraic group when the characteristic of k is zero or very good for G (for the definition of “very good characteristic” see [Car85, §1.14]).

Fix a Borel subgroup B in G and a maximal torus T in B . Define U to be the unipotent radical of B and define $W = N_G(T)/T$ to be the Weyl group of (G, T) . Set $n = \dim \mathcal{B}$ and $r = \dim T$.

We will use the convention that a lowercase fraktur letter denotes the Lie algebra of the algebraic group denoted by the corresponding uppercase roman letter.

For x in \mathfrak{N} , define $\mathcal{B}_x = \{gBg^{-1} \mid g^{-1}x \in \mathfrak{b}\}$, the *Springer fibre* at x .

2.1. Irreducible components of Z , Weyl group elements, and nilpotent orbits. We begin analyzing the geometry of Z using ideas that go back to Steinberg [Ste76] and Spaltenstein [Spa82].

The group G acts on \mathcal{B} by conjugation and on \mathfrak{N} by the adjoint action. This latter action is denoted by $(g, x) \mapsto g \cdot x = gx$. Thus, G acts “diagonally” on Z .

Let $\pi: Z \rightarrow \mathcal{B} \times \mathcal{B}$ be the projection on the second and third factors. By the Bruhat Lemma, the elements of W parametrize the G -orbits on $\mathcal{B} \times \mathcal{B}$. An element w in W corresponds to the G -orbit containing (B, wBw^{-1}) in $\mathcal{B} \times \mathcal{B}$. Define

$$Z_w = \pi^{-1}(G(B, wBw^{-1})), \quad U_w = U \cap wUw^{-1}, \quad \text{and} \quad B_w = B \cap wBw^{-1}.$$

The varieties Z_w play a key role in the rest of this paper.

For w in W , the restriction of π to Z_w is a G -equivariant morphism from Z_w onto a transitive G -space. The fibre over the point (B, wBw^{-1}) is isomorphic to \mathfrak{u}_w and so it follows from [Slo80, II 3.7] that Z_w is isomorphic to the associated fibre bundle $G \times^{B_w} \mathfrak{u}_w$. Thus, Z_w is irreducible and $\dim Z_w = \dim G - \dim B_w + \dim \mathfrak{u}_w = 2n$. Furthermore, each Z_w is locally closed in Z and so it follows that $\{\overline{Z_w} \mid w \in W\}$ is the set of irreducible components of Z .

Now let $\mu_z: Z \rightarrow \mathfrak{N}$ denote the projection on the first component. For a G -orbit, \mathfrak{O} , in \mathfrak{N} , set $Z_{\mathfrak{O}} = \mu_z^{-1}(\mathfrak{O})$ and fix x in \mathfrak{O} . Then the restriction of μ_z to $Z_{\mathfrak{O}}$ is a G -equivariant morphism from $Z_{\mathfrak{O}}$ onto a transitive G -space. The fibre over x is isomorphic to $\mathcal{B}_x \times \mathcal{B}_x$ and so it follows from [Slo80, II 3.7] that $Z_{\mathfrak{O}} \cong G \times^{Z_G(x)} (\mathcal{B}_x \times \mathcal{B}_x)$. Spaltenstein [Spa82, §II.1] has shown that the variety \mathcal{B}_x is equidimensional and Steinberg and Spaltenstein have shown that $\dim Z_G(x) = r + 2 \dim \mathcal{B}_x$. This implies the following results due to Steinberg [Ste76, Proposition 3.1]:

- (1) $\dim Z_{\mathfrak{O}} = \dim G - \dim Z_G(x) + 2 \dim \mathcal{B}_x = \dim G - r = 2n$.
- (2) Every irreducible component of $Z_{\mathfrak{O}}$ has the form

$$G(\{x\} \times C_1 \times C_2) = G(\{x\} \times (Z_G(x)(C_1 \times C_2)))$$

where C_1 and C_2 are irreducible components of \mathcal{B}_x .

- (3) A pair, (C'_1, C'_2) , of irreducible components of \mathcal{B}_x determines the same irreducible component of $Z_{\mathfrak{O}}$ as (C_1, C_2) if and only if there is a z in $Z_G(x)$ with $(C'_1, C'_2) = (zC_1z^{-1}, zC_2z^{-1})$.

From (2) we see that $Z_{\mathfrak{D}}$ is equidimensional with $\dim Z_{\mathfrak{D}} = 2n = \dim Z$ and from (3) we see that there is a bijection between irreducible components of $Z_{\mathfrak{D}}$ and $Z_G(x)$ -orbits on the set of irreducible components of $\mathcal{B}_x \times \mathcal{B}_x$.

The closures of the irreducible components of $Z_{\mathfrak{D}}$ are closed, irreducible, $2n$ -dimensional subvarieties of Z and so each irreducible component of $Z_{\mathfrak{D}}$ is of the form $Z_{\mathfrak{D}} \cap \overline{Z_w}$ for some unique w in W . Define $W_{\mathfrak{D}}$ to be the subset of W that parametrizes the irreducible components of $Z_{\mathfrak{D}}$. Then w is in $W_{\mathfrak{D}}$ if and only if $Z_{\mathfrak{D}} \cap \overline{Z_w}$ is an irreducible component of $Z_{\mathfrak{D}}$.

Clearly, W is the disjoint union of the $W_{\mathfrak{D}}$'s as \mathfrak{D} varies over the nilpotent orbits in \mathfrak{N} . The subsets $W_{\mathfrak{D}}$ are called *two-sided Steinberg cells*. Two-sided Steinberg cells have several properties in common with two-sided Kazhdan-Lusztig cells in W . Some of the properties of two-sided Steinberg cells will be described in the next subsection. Kazhdan-Lusztig cells were introduced in [KL79, §1]. We will briefly review this theory in §4.4.

In general there are more two-sided Steinberg cells than two-sided Kazhdan-Lusztig cells. This may be seen as follows. Clearly, two-sided Steinberg cells are in bijection with the set of G -orbits in \mathfrak{N} .

Two-sided Kazhdan-Lusztig cells may be related to nilpotent orbits through the Springer correspondence using Lusztig's analysis of Kazhdan-Lusztig cells in Weyl groups. We will review the Springer correspondence in §3.4 below, where we will see that there is an injection from the set of nilpotent orbits to the set of irreducible representations of W given by associating with \mathfrak{D} the representation of W on $H_{2d_x}(\mathcal{B}_x)^{C(x)}$, where x is in \mathfrak{D} and $C(x)$ is the component group of x . Two-sided Kazhdan-Lusztig cells determine a filtration of the group algebra $\mathbb{Q}[W]$ by two-sided ideals (see §4.4) and in the associated graded $W \times W$ -module, each summand contains a distinguished representation that is called *special* (see [Lus79] and [Lus84, Chapter 5]). The case-by-case computation of the Springer correspondence shows that every special representation of W is equivalent to the representation of W on $H_{2d_x}(\mathcal{B}_x)^{C(x)}$ for some x . The resulting nilpotent orbits are called *special* nilpotent orbits.

If G has type A_l , then every irreducible representation of W and every nilpotent orbit is special but otherwise there are non-special irreducible representation of W and nilpotent orbits. Although in general there are fewer two-sided Kazhdan-Lusztig cells in W than two-sided Steinberg cells, Lusztig [Lus89b, §4] has constructed a bijection between the set of two-sided Kazhdan-Lusztig cells in the extended, affine, Weyl group, W_e , and the set of G -orbits in \mathfrak{N} . Thus, there is a bijection between two-sided Steinberg cells in W and two-sided Kazhdan-Lusztig cells in W_e . We will describe this bijection in §4.4 in connection with the computation of the equivariant K -theory of the Steinberg variety.

Suppose \mathfrak{D} is a nilpotent orbit and x is in \mathfrak{D} . We can explicitly describe the bijection in (c) above between $W_{\mathfrak{D}}$ and the $Z_G(x)$ -orbits on the set of pairs of irreducible components of \mathcal{B}_x as follows. If w is in $W_{\mathfrak{D}}$ and (C_1, C_2) is a pair of irreducible components of \mathcal{B}_x , then w corresponds to the $Z_G(x)$ -orbit of (C_1, C_2) if and only if $G(B, wBw^{-1}) \cap (C_1 \times C_2)$ is dense in $C_1 \times C_2$.

Using the isomorphism $Z_w \cong G \times^{B_w} \mathfrak{u}_w$ we see that $Z_{\mathfrak{D}} \cap Z_w \cong G \times^{B_w} (\mathfrak{D} \cap \mathfrak{u}_w)$. Therefore, w is in $W_{\mathfrak{D}}$ if and only if $\mathfrak{D} \cap \mathfrak{u}_w$ is dense in \mathfrak{u}_w . This shows in particular that $W_{\mathfrak{D}}$ is closed under taking inverses.

We conclude this subsection with some examples of two-sided Steinberg cells.

When $x = 0$ we have $Z_{\{0\}} = \overline{Z_{w_0}} = \{0\} \times \mathcal{B} \times \mathcal{B}$ where w_0 is the longest element in W . Therefore, $W_{\{0\}} = \{w_0\}$.

At the other extreme, let $\mathfrak{N}_{\text{reg}}$ denote the regular nilpotent orbit. Then it follows from the fact that every regular nilpotent element is contained in a unique Borel subalgebra that $W_{\mathfrak{N}_{\text{reg}}}$ contains just the identity element in W .

For G of type A_l , it follows from a result of Spaltenstein [Spa76] that two elements of W lie in the same two-sided Steinberg cell if and only if they yield the same Young diagram under the Robinson-Schensted correspondence. A more refined result due to Steinberg will be discussed at the end of the next subsection.

2.2. Orbital varieties. Suppose that \mathfrak{D} is a nilpotent orbit. An *orbital variety* for \mathfrak{D} is an irreducible component of $\mathfrak{D} \cap \mathfrak{u}$. An *orbital variety* is a subvariety of \mathfrak{N} that is orbital for some nilpotent orbit. The reader should be aware that sometimes an orbital variety is defined as the closure of an irreducible component of $\mathfrak{D} \cap \mathfrak{u}$.

We will see in this subsection that orbital varieties can be used to decompose two-sided Steinberg cells into left and right Steinberg cells and to refine the relationship between nilpotent orbits and elements of W . When G is of type A_l and W is the symmetric group S_{l+1} , the decomposition of a two-sided Steinberg cell into left and right Steinberg cells can be viewed as a geometric realization of the Robinson-Schensted correspondence.

We will see in the next subsection that orbital varieties arise in the theory of associated varieties of finitely generated \mathfrak{g} -modules.

Fix a nilpotent orbit \mathfrak{D} and an element x in $\mathfrak{D} \cap \mathfrak{u}$. Define $p: G \rightarrow \mathfrak{D}$ by $p(g) = g^{-1}x$ and $q: G \rightarrow \mathcal{B}$ by $q(g) = gBg^{-1}$. Then $p^{-1}(\mathfrak{D} \cap \mathfrak{u}) = q^{-1}(\mathcal{B}_x)$. Spaltenstein [Spa82, §II.2] has shown that

- (1) if C is an irreducible component of \mathcal{B}_x , then $pq^{-1}(C)$ is an orbital variety for \mathfrak{D} ,
- (2) every orbital variety for \mathfrak{D} has the form $pq^{-1}(C)$ for some irreducible component C of \mathcal{B}_x , and
- (3) $pq^{-1}(C) = pq^{-1}(C')$ for components C and C' of \mathcal{B}_x if and only if C and C' are in the same $Z_G(x)$ -orbit.

It follows immediately that $\mathfrak{D} \cap \mathfrak{u}$ is equidimensional and all orbital varieties for \mathfrak{D} have the same dimension: $n - \dim \mathcal{B}_x = \frac{1}{2} \dim \mathfrak{D}$.

We decompose two-sided Steinberg cells into left and right Steinberg cells following a construction of Joseph [Jos84, §9].

Suppose \mathfrak{V}_1 and \mathfrak{V}_2 are orbital varieties for \mathfrak{D} . Choose irreducible components C_1 and C_2 of \mathcal{B}_x so that $pq^{-1}(C_1) = \mathfrak{V}_1$ and $pq^{-1}(C_2) = \mathfrak{V}_2$. We have seen that there is a w in $W_{\mathfrak{D}}$ so that $Z_{\mathfrak{D}} \cap \overline{Z_w} = G(\{x\} \times Z_G(x)(C_1 \times C_2))$. Clearly, $\overline{\mu_z^{-1}(x) \cap Z_w} \subseteq \mu_z^{-1}(x) \cap \overline{Z_w}$. Since both sides are closed, both sides are $Z_G(x)$ -stable, and the right hand side is the $Z_G(x)$ -saturation of $\{x\} \times C_1 \times C_2$, it follows that $\overline{\mu_z^{-1}(x) \cap Z_w} = \mu_z^{-1}(x) \cap \overline{Z_w}$.

Let p_2 denote the projection of $Z_{\mathfrak{D}}$ to \mathcal{B} by $p_2(x, B', B'') = B'$. Then $pq^{-1}p_2(\mu_z^{-1}(x) \cap Z_w) = B(\mathfrak{D} \cap \mathfrak{u}_w)$. Also,

$$pq^{-1}p_2(\mu_z^{-1}(x) \cap \overline{Z_w}) = pq^{-1}p_2(\{x\} \times Z_G(x)(C_1 \times C_2)) = pq^{-1}(Z_G(x)C_1) = \mathfrak{V}_1.$$

Since $\mathfrak{D} \cap \mathfrak{u}_w$ is dense in \mathfrak{u}_w we have $\overline{B\mathfrak{u}_w} \cap \mathfrak{D} = \overline{B(\mathfrak{D} \cap \mathfrak{u}_w)} \subseteq \mathfrak{V}_1$. However, since $\mu_z^{-1}(x) \cap Z_w$ is a dense, $Z_G(x)$ -stable subset of $\mu_z^{-1}(x) \cap \overline{Z_w}$, it follows that

$$\dim B(\mathfrak{D} \cap \mathfrak{u}_w) = \dim pq^{-1}p_2(\mu_z^{-1}(x) \cap Z_w)$$

$$\begin{aligned}
&= \dim p_2(\mu_z^{-1}(x) \cap \overline{Z_w}) + \dim B - \dim Z_G(x) \\
&= \dim \mathcal{B}_x + \dim B - r - 2 \dim \mathcal{B}_x \\
&= n - \dim \mathcal{B}_x
\end{aligned}$$

and so $\overline{Bu_w} \cap \mathfrak{D} = \mathfrak{V}_1$.

A similar argument shows that $\overline{Bu_{w^{-1}}} \cap \mathfrak{D} = \mathfrak{V}_2$. This proves the following theorem.

Theorem 2.1. *If \mathfrak{D} is a nilpotent orbit and \mathfrak{V}_1 and \mathfrak{V}_2 are orbital varieties for \mathfrak{D} , then there is a w in $W_{\mathfrak{D}}$ so that $\mathfrak{V}_1 = \overline{Bu_w} \cap \mathfrak{D}$ and $\mathfrak{V}_2 = \overline{Bu_{w^{-1}}} \cap \mathfrak{D}$.*

Conversely, if w is in $W_{\mathfrak{D}}$, then $\overline{Bu_w}$ is irreducible and the arguments above show that $\mathfrak{u}_w \cap \mathfrak{D}$ is dense in \mathfrak{u}_w and then that $\overline{Bu_w} \cap \mathfrak{D}$ is an orbital variety. This proves the next proposition.

Proposition 2.2. *Orbital varieties are the subsets of \mathfrak{u} of the form $\overline{Bu_w} \cap \mathfrak{D}$, where $\mathfrak{u}_w \cap \mathfrak{D}$ is dense in \mathfrak{u}_w .*

For w in W , define $\mathfrak{V}_l(w) = \overline{Bu_{w^{-1}}} \cap \mathfrak{D}$ when w is in $W_{\mathfrak{D}}$. For w_1 and w_2 in W , define $w_1 \sim_l w_2$ if $\mathfrak{V}_l(w_1) = \mathfrak{V}_l(w_2)$. Then \sim_l is an equivalence relation and the equivalence classes are called *left Steinberg cells*. Similarly, define $\mathfrak{V}_r(w) = \overline{Bu_w} \cap \mathfrak{D}$ when w is in $W_{\mathfrak{D}}$ and $w_1 \sim_r w_2$ if $\mathfrak{V}_r(w_1) = \mathfrak{V}_r(w_2)$. The equivalence classes for \sim_r are called *right Steinberg cells*.

Clearly, each two-sided Steinberg cell is a disjoint union of left Steinberg cells and is also the disjoint union of right Steinberg cells. Precisely, if w is in $W_{\mathfrak{D}}$, then

$$W_{\mathfrak{D}} = \coprod_{y \in \mathfrak{V}_r(w)} \mathfrak{V}_l(y) = \coprod_{y \in \mathfrak{V}_l(w)} \mathfrak{V}_r(y).$$

It follows from Theorem 2.1 that the rule $w \mapsto (\mathfrak{V}_r(w), \mathfrak{V}_l(w))$ defines a surjection from W to the set of pairs of orbital varieties for the same nilpotent orbit. We will see in §3.4 that the number of orbital varieties for a nilpotent orbit \mathfrak{D} is the dimension of the Springer representation of W corresponding to the trivial representation of the component group of any element in \mathfrak{D} . Denote this representation of W by $\rho_{\mathfrak{D}}$. Then the number of pairs $(\mathfrak{V}_1, \mathfrak{V}_2)$, where \mathfrak{V}_1 and \mathfrak{V}_2 are orbital varieties for the same nilpotent orbit, is $\sum_{\mathfrak{D}} (\dim \rho_{\mathfrak{D}})^2$. In general this sum is strictly smaller than $|W|$. Equivalently, in general, there are more irreducible representations of W than G -orbits in \mathfrak{N} .

However, if G has type A , for example if $G = \mathrm{SL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{C})$, then every irreducible representation of W is of the form $\rho_{\mathfrak{D}}$ for a unique nilpotent orbit \mathfrak{D} . In this case $w \mapsto (\mathfrak{V}_r(w), \mathfrak{V}_l(w))$ defines a bijection from W to the set of pairs of orbital varieties for the same nilpotent orbit. Steinberg has shown that this bijection is essentially given by the Robinson-Schensted correspondence.

In more detail, using the notation in the proof of Theorem 2.1, suppose that \mathfrak{D} is a nilpotent orbit, \mathfrak{V}_1 and \mathfrak{V}_2 are orbital varieties for \mathfrak{D} , and C_1 and C_2 are the corresponding irreducible components in \mathcal{B}_x . In [Ste88] Steinberg defines a function from \mathcal{B} to the set of standard Young tableaux and shows that $G(B, wBw^{-1}) \cap (C_1 \times C_2)$ is dense in $C_1 \times C_2$ if and only if the pair of standard Young tableaux associated to a generic pair (B', B'') in $C_1 \times C_2$ is the same as the pair of standard Young tableaux associated to w by the Robinson-Schensted correspondence. For more details, see also [Dou96].

An open problem, even in type A , is determining the orbit closures of orbital varieties. Some rudimentary information may be obtained by considering the top Borel-Moore homology group of Z (see §3 below and [HJ05, §4, §5]).

2.3. Associated varieties and characteristic varieties. The Steinberg variety and orbital varieties also arise naturally in the Beilinson-Bernstein theory of algebraic (\mathcal{D}, K) -modules [BB81]. This was first observed by Borho and Brylinski [BB85] and Ginzburg [Gin86]. In this subsection we begin with a review of the Beilinson-Bernstein Localization Theorem and its connection with the computation of characteristic varieties and associated varieties. Then we describe an equivariant version of this theory. It is in the equivariant theory that the Steinberg variety naturally occurs.

For a variety X (over \mathbb{C}), let \mathcal{O}_X denote the structure sheaf of X , $\mathbb{C}[X] = \Gamma(X, \mathcal{O}_X)$ the algebra of global, regular functions on X , and \mathcal{D}_X the sheaf of algebraic differential operators on X . On an open subvariety, V , of X , $\Gamma(V, \mathcal{D}_X)$ is the subalgebra of $\text{Hom}_{\mathbb{C}}(\mathbb{C}[V], \mathbb{C}[V])$ generated by multiplication by elements of $\mathbb{C}[V]$ and \mathbb{C} -linear derivations of $\mathbb{C}[V]$. Define $D_X = \Gamma(X, \mathcal{D}_X)$, the algebra of global, algebraic, differential operators on X .

A *quasi-coherent \mathcal{D}_X -module* is a left \mathcal{D}_X -module that is quasi-coherent when considered as an \mathcal{O}_X -module. Generalizing a familiar result for affine varieties, Beilinson-Bernstein [BB81, §2] have proved that for $X = \mathcal{B}$, the global section functor, $\Gamma(\mathcal{B}, \cdot)$, defines an equivalence of categories between the category of quasi-coherent $\mathcal{D}_{\mathcal{B}}$ -modules and the category of $D_{\mathcal{B}}$ -modules.

In turn, the algebra $D_{\mathcal{B}}$ is isomorphic to $U(\mathfrak{g})/I_0$, where $U(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} and I_0 denotes the two-sided ideal in $U(\mathfrak{g})$ generated by the kernel of the trivial character of the center of $U(\mathfrak{g})$ (see [BB82, §3]). Thus, the category of $D_{\mathcal{B}}$ -modules is equivalent to the category of $U(\mathfrak{g})/I_0$ -modules, that is, the category of $U(\mathfrak{g})$ -modules with trivial central character.

Composing these two equivalences we see that the category of quasi-coherent $\mathcal{D}_{\mathcal{B}}$ -modules is equivalent to the category of $U(\mathfrak{g})$ -modules with trivial central character. In this equivalence, coherent $\mathcal{D}_{\mathcal{B}}$ -modules (that is, $\mathcal{D}_{\mathcal{B}}$ -modules that are coherent when considered as $\mathcal{O}_{\mathcal{B}}$ -modules) correspond to finitely generated $U(\mathfrak{g})$ -modules with trivial central character.

The equivalence of categories between coherent $\mathcal{D}_{\mathcal{B}}$ -modules and finitely generated $U(\mathfrak{g})$ -modules with trivial central character has a geometric shadow that can be described using the “moment map” of the G -action on the cotangent bundle of \mathcal{B} .

Let B' be a Borel subgroup of G . Then using the Killing form on \mathfrak{g} , the cotangent space to \mathcal{B} at B' may be identified with $\mathfrak{b}' \cap \mathfrak{N}$, the nilradical of \mathfrak{b}' . Define

$$\tilde{\mathfrak{N}} = \{ (x, B') \in \mathfrak{N} \times \mathcal{B} \mid x \in \mathfrak{b}' \}$$

and let $\mu: \tilde{\mathfrak{N}} \rightarrow \mathfrak{N}$ be the projection on the first factor. Then $\tilde{\mathfrak{N}} \cong T^*\mathcal{B}$, the cotangent bundle of \mathcal{B} . It is easy to see that $Z \cong \tilde{\mathfrak{N}} \times_{\mathfrak{N}} \tilde{\mathfrak{N}} \cong T^*\mathcal{B} \times_{\mathfrak{N}} T^*\mathcal{B}$.

Using the orders of differential operators, we obtain a filtration of \mathcal{D}_X . With respect to this filtration, the associated graded sheaf $\text{gr } \mathcal{D}_{\mathcal{B}}$ is isomorphic to the direct image $p_* \mathcal{O}_{T^*\mathcal{B}}$, where $p: T^*\mathcal{B} \rightarrow \mathcal{B}$ is the projection.

Let \mathcal{M} be a coherent $\mathcal{D}_{\mathcal{B}}$ -module. Then \mathcal{M} has a “good” filtration such that $\text{gr } \mathcal{M}$ is a coherent $\text{gr } \mathcal{D}_{\mathcal{B}}$ -module. Since $\text{gr } \mathcal{D}_{\mathcal{B}} \cong p_* \mathcal{O}_{T^*\mathcal{B}}$, we see that $\text{gr } \mathcal{M}$ has the structure of a coherent $\mathcal{O}_{T^*\mathcal{B}}$ -module. The *characteristic variety* of \mathcal{M} is the support in $T^*\mathcal{B}$ of the $\mathcal{O}_{T^*\mathcal{B}}$ -module $\text{gr } \mathcal{M}$. Using the isomorphism $T^*\mathcal{B} \cong \tilde{\mathfrak{N}}$, we identify the characteristic variety of \mathcal{M} with a closed subvariety of $\tilde{\mathfrak{N}}$ and denote this latter variety by $V_{\tilde{\mathfrak{N}}}(\mathcal{M})$. It is known that $V_{\tilde{\mathfrak{N}}}(\mathcal{M})$ is independent of the choice of good filtration.

Now consider the enveloping algebra $U(\mathfrak{g})$ with the standard filtration. By the PBW Theorem, $\text{gr } U(\mathfrak{g}) \cong \text{Sym}(\mathfrak{g})$, the symmetric algebra of \mathfrak{g} . Using the Killing form, we identify \mathfrak{g} with its linear dual, \mathfrak{g}^* , and $\text{gr } U(\mathfrak{g})$ with $\mathbb{C}[\mathfrak{g}]$. Let M be a finitely generated $U(\mathfrak{g})$ -module. Then M has a “good” filtration such that the associated graded module, $\text{gr } M$, a module for $\text{gr } U(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}]$, is finitely generated. The *associated variety* of M , denoted by $V_{\mathfrak{g}}(M)$, is the support of the $\mathbb{C}[\mathfrak{g}]$ -module $\text{gr } M$ – a closed subvariety of \mathfrak{g} . It is known that $V_{\mathfrak{g}}(M)$ is independent of the choice of good filtration.

Borho and Brylinski [BB85, §1.9] have proved the following theorem.

Theorem 2.3. *Suppose that \mathcal{M} is a coherent $\mathcal{D}_{\mathcal{B}}$ -module and let M denote the space of global sections of \mathcal{M} . Then $V_{\mathfrak{g}}(M) \subseteq \mathfrak{N}$ and $\mu(V_{\widetilde{\mathfrak{N}}}(\mathcal{M})) = V_{\mathfrak{g}}(M)$.*

There are equivariant versions of the above constructions which incorporate a subgroup of G that acts on \mathcal{B} with finitely many orbits. It is in this equivariant context that the Steinberg variety and orbital varieties make their appearance.

Suppose that K is a closed, connected, algebraic subgroup of G that acts on \mathcal{B} with finitely many orbits. The two special cases we are interested in are the “highest weight” case, when $K = B$ is a Borel subgroup of G , and the “Harish-Chandra” case, when $K = G_d$ is the diagonal subgroup of $G \times G$.

In the general setting, we suppose that W is a finite set that indexes the K -orbits on \mathcal{B} by $w \leftrightarrow X_w$. Of course, in the examples we are interested in, we know that the Weyl group W indexes the set of orbits of K on \mathcal{B} .

For w in W , let $T_w^* \mathcal{B}$ denote the conormal bundle to the K -orbit X_w in $T^* \mathcal{B}$. Then letting \mathfrak{k}^\perp denote the subspace of \mathfrak{g} orthogonal to \mathfrak{k} with respect to the Killing form and using our identification of $T^* \mathcal{B}$ with pairs, we may identify

$$T_w^* \mathcal{B} = \{ (x, B') \in \mathfrak{N} \times \mathcal{B} \mid B' \in X_w, x \in \mathfrak{b}' \cap \mathfrak{k}^\perp \}.$$

Define $Y_{\mathfrak{k}^\perp} = \mu^{-1}(\mathfrak{k}^\perp \cap \mathfrak{N})$. Then $Y_{\mathfrak{k}^\perp}$ is closed, $Y_{\mathfrak{k}^\perp} = \coprod_{w \in W} T_w^* \mathcal{B} = \cup_{w \in W} \overline{T_w^* \mathcal{B}}$, and μ restricts to a surjection $Y_{\mathfrak{k}^\perp} \xrightarrow{\mu} \mathfrak{k}^\perp$ (see [BB85, §2.4]). Summarizing, we have a commutative diagram

$$(2.4) \quad \begin{array}{ccc} Y_{\mathfrak{k}^\perp} & \longrightarrow & \widetilde{\mathfrak{N}} \\ \mu \downarrow & & \downarrow \mu \\ \mathfrak{k}^\perp \cap \mathfrak{N} & \longrightarrow & \mathfrak{N} \end{array}$$

where the horizontal arrows are inclusions. Moreover, for w in W , $\dim T_w^* \mathcal{B} = \dim \mathcal{B}$ and $T_w^* \mathcal{B}$ is locally closed in $Y_{\mathfrak{k}^\perp}$. Thus, the set of irreducible components of $Y_{\mathfrak{k}^\perp}$ is $\{ \overline{T_w^* \mathcal{B}} \mid w \in W \}$.

A *quasi-coherent* $(\mathcal{D}_{\mathcal{B}}, K)$ -module is a K -equivariant, quasi-coherent $\mathcal{D}_{\mathcal{B}}$ -module (for the precise definition see [BB85, §2]). If \mathcal{M} is a coherent $(\mathcal{D}_{\mathcal{B}}, K)$ -module, then $V_{\widetilde{\mathfrak{N}}}(\mathcal{M}) \subseteq Y_{\mathfrak{k}^\perp}$.

Similarly, a (\mathfrak{g}, K) -module is a \mathfrak{g} -module with a compatible algebraic action of K (for the precise definition see [BB85, §2]). If M is a finitely generated (\mathfrak{g}, K) -module, then $V_{\mathfrak{g}}(M)$ is contained in \mathfrak{k}^\perp .

As in the non-equivariant setting, Beilinson-Bernstein [BB81, §2] have proved that the global section functor, $\Gamma(\mathcal{B}, \cdot)$, defines an equivalence of categories between the category of quasi-coherent $(\mathcal{D}_{\mathcal{B}}, K)$ -modules and the category of (\mathfrak{g}, K) -modules with trivial central character. Under this equivalence, coherent $(\mathcal{D}_{\mathcal{B}}, K)$ -modules correspond to finitely generated (\mathfrak{g}, K) -modules with trivial central character.

The addition of a K -action results in a finer version of Theorem 2.3 (see [BB85, §4]).

Theorem 2.5. *Suppose that \mathcal{M} is a coherent $(\mathcal{D}_{\mathcal{B}}, K)$ -module and let M denote the space of global sections of \mathcal{M} .*

- (a) *The variety $V_{\tilde{\mathfrak{N}}}(\mathcal{M})$ is a union of irreducible components of $Y_{\mathfrak{k}^\perp}$ and so there is a subset $\Sigma(\mathcal{M})$ of W such that $V_{\tilde{\mathfrak{N}}}(\mathcal{M}) = \bigcup_{w \in \Sigma(\mathcal{M})} \overline{T_w^* \mathcal{B}}$.*
- (b) *The variety $V_{\mathfrak{g}}(M)$ is contained in $\mathfrak{k}^\perp \cap \mathfrak{N}$ and*

$$V_{\mathfrak{g}}(M) = \mu(V_{\tilde{\mathfrak{N}}}(\mathcal{M})) = \bigcup_{w \in \Sigma(\mathcal{M})} \mu(\overline{T_w^* \mathcal{B}}).$$

Now it is time to unravel the notation in the highest weight and Harish-Chandra cases.

First consider the highest weight case when $K = B$. We have $\mathfrak{k}^\perp = \mathfrak{b}^\perp = \mathfrak{u}$. Hence, $Y_{\mathfrak{u}^\perp} = \mu^{-1}(\mathfrak{u}) \cong \{(x, B') \in \mathfrak{N} \times \mathcal{B} \mid x \in \mathfrak{u} \cap \mathfrak{b}'\}$. We denote $Y_{\mathfrak{u}^\perp}$ simply by Y and call it the *conormal variety*. For w in W , X_w is the set of B -conjugates of wBw^{-1} and $T_w^* \mathcal{B} \cong \{(x, B') \in \mathfrak{N} \times \mathcal{B} \mid B' \in X_w, x \in \mathfrak{u} \cap \mathfrak{b}'\}$. The projection of $T_w^* \mathcal{B}$ to \mathcal{B} is a B -equivariant surjection onto X_w and so $T_w^* \mathcal{B} \cong B \times^{B_w} \mathfrak{u}_w$. The diagram (2.4) becomes

$$\begin{array}{ccc} Y & \longrightarrow & \tilde{\mathfrak{N}} \\ \mu \downarrow & & \downarrow \mu \\ \mathfrak{u} & \longrightarrow & \mathfrak{N}. \end{array}$$

Moreover, for w in W , $\mu(T_w^* \mathcal{B}) = B\mathfrak{u}_w$. Since μ is proper, it follows that $\mu(\overline{T_w^* \mathcal{B}}) = \overline{B\mathfrak{u}_w}$ is the closure of an orbital variety.

Arguments in the spirit of those given in §2.1 (see [HJ05, §3]) show that if we set $Y_w = T_w^* \mathcal{B}$ and $Y_{\mathfrak{D}} = \mu^{-1}(\mathfrak{D} \cap \mathfrak{u})$, then $\dim Y_{\mathfrak{D}} = n$, $\overline{Y_{\mathfrak{D}}}$ is equidimensional, and the set of irreducible components of $\overline{Y_{\mathfrak{D}}}$ is $\{\overline{Y_{\mathfrak{D}} \cap Y_w} \mid w \in W_{\mathfrak{D}}\}$.

Next consider the Harish-Chandra case. In this setting, the ambient group is $G \times G$ and $K = G_d$ is the diagonal subgroup. Clearly, $\mathfrak{k}^\perp = \mathfrak{g}_d^\perp = \{(x, -x) \mid x \in \mathfrak{g}\}$ is isomorphic to \mathfrak{g} and so

$$Y_{\mathfrak{g}_d^\perp} = (\mu \times \mu)^{-1}(\mathfrak{g}_d^\perp) = \{(x, -x, B', B'') \in \mathfrak{g} \times \mathfrak{g} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{b}' \cap \mathfrak{b}'' \cap \mathcal{N}\}.$$

Thus, in this case, $Y_{\mathfrak{g}_d^\perp}$ is clearly isomorphic to the Steinberg variety and we may identify the restriction of $\mu \times \mu$ to $Y_{\mathfrak{g}_d^\perp}$ with $\mu_z: Z \rightarrow \mathfrak{N}$. The diagram (2.4) becomes

$$\begin{array}{ccc} Z & \longrightarrow & \tilde{\mathfrak{N}} \times \tilde{\mathfrak{N}} \\ \mu_z \downarrow & & \downarrow \mu \times \mu \\ \mathfrak{N} & \longrightarrow & \mathfrak{N} \times \mathfrak{N} \end{array}$$

where the bottom horizontal map is given by $x \mapsto (x, -x)$. Moreover, for w in W ,

$$T_w^*(\mathcal{B} \times \mathcal{B}) = \{(x, -x, B', B'') \mid (B', B'') \in G(B, wBw^{-1}), x \in \mathfrak{b}' \cap \mathfrak{b}'' \cap \mathcal{N}\} \cong Z_w.$$

Let $p_3: Z \rightarrow \mathcal{B}$ be the projection on the third factor. Then p_3 is G -equivariant, G acts transitively on \mathcal{B} , and the fibre over B is isomorphic to Y . This gives yet another description of the Steinberg variety: $Z \cong G \times^B Y$.

Now consider the following three categories:

- coherent $(\mathcal{D}_{\mathcal{B} \times \mathcal{B}}, G_d)$ -modules, $\text{Mod}(\mathcal{D}_{\mathcal{B} \times \mathcal{B}}, G_d)^{\text{coh}}$;

- finitely generated $(\mathfrak{g} \times \mathfrak{g}, G_d)$ -modules with trivial central character, $\text{Mod}(\mathfrak{g} \times \mathfrak{g}, G_d)_{0,0}^{\text{fg}}$; and
- finitely generated (\mathfrak{g}, B) -modules with trivial central character, $\text{Mod}(\mathfrak{g}, B)_0^{\text{fg}}$.

We have seen that the global section functor defines an equivalence of categories between $\text{Mod}(\mathcal{D}_{\mathcal{B} \times \mathcal{B}}, G_d)^{\text{coh}}$ and $\text{Mod}(\mathfrak{g} \times \mathfrak{g}, G_d)_{0,0}^{\text{fg}}$. Bernstein and Gelfand [BG80], as well as Joseph [Jos79], have constructed an equivalence of categories between $\text{Mod}(\mathfrak{g} \times \mathfrak{g}, G_d)_{0,0}^{\text{fg}}$ and $\text{Mod}(\mathfrak{g}, B)_0^{\text{fg}}$.

Composing these two equivalences of categories we see that the category of coherent $(\mathcal{D}_{\mathcal{B} \times \mathcal{B}}, G_d)$ -modules is equivalent to the category of finitely generated (\mathfrak{g}, B) -modules with trivial central character, $\text{Mod}(\mathfrak{g}, B)_0^{\text{fg}}$. Both equivalences behave well with respect to characteristic varieties and associated varieties and hence so does their composition. This is the content of the next theorem. The theorem extends Theorem 2.5 and summarizes the relationships between the various constructions in this subsection. See [BB85, §4] for the proof.

Theorem 2.6. *Suppose \mathcal{M} is a coherent $(\mathcal{D}_{\mathcal{B} \times \mathcal{B}}, G_d)$ -module, M is the space of global sections of \mathcal{M} , and L is the finitely generated (\mathfrak{g}, B) -module with trivial central character corresponding to M . Let $\Sigma = \Sigma(\mathcal{M})$ be as in Theorem 2.5. Then when $\mu \times \mu: Y_{\mathfrak{g}_d^\perp} \rightarrow \mathfrak{g}_d^\perp$ is identified with $\mu_z: Z \rightarrow \mathfrak{N}$ we have:*

- The characteristic variety of \mathcal{M} is $V_{T^*(\mathcal{B} \times \mathcal{B})}(\mathcal{M}) = \cup_{y \in \Sigma} \overline{Z_y}$, a union of irreducible components of the Steinberg variety.*
- The associated variety of M is $V_{\mathfrak{g}}(M) = \mu_z(V_{\mathfrak{g}}(\mathcal{M})) = \cup_{y \in \Sigma} \overline{G\mathbf{u}_y} = G \cdot V_{\mathfrak{u}}(L)$, so the associated variety of M is the image under μ_z of the characteristic variety of \mathcal{M} and is also the G -saturation of the associated variety of L .*
- The associated variety of L is $V_{\mathfrak{u}}(L) = \cup_{y \in \Sigma} \overline{B\mathbf{u}_y}$, a union of closures of orbital varieties.*

The characteristic variety of a coherent $(\mathcal{D}_{\mathcal{B} \times \mathcal{B}}, G_d)$ -module is the union of the characteristic varieties of its composition factors. Similarly the associated variety of a finitely generated $(\mathfrak{g} \times \mathfrak{g}, G_d)$ -module or a finitely generated (\mathfrak{g}, B) -module depends only on its composition factors. Thus, computing characteristic and associated varieties reduces to the case of simple modules. The simple objects in each of these categories are indexed by W , see [BB81, §3] and [BB85, §2.7, 4.3, 4.8]. If w is in W_{Σ} and \mathcal{M}_w , M_w , and L_w are corresponding simple modules, then it is shown in [BB85, §4.9] that $\mu_z(V_{\mathfrak{g}}(\mathcal{M}_w)) = V(M_w) = G \cdot V(L_w) = \overline{\Sigma}$.

In general, explicitly computing the subset $\Sigma = \Sigma(\mathcal{M}_w)$ so that $V_Z(\mathcal{M}_w) = \cup_{y \in \Sigma} \overline{Z_y}$ and $V_{\mathfrak{u}}(L_w) = \cup_{y \in \Sigma} \overline{B\mathbf{u}_y}$ for w in W is a very difficult and open problem. See [BB85, §4.3] and [HJ05, §6] for examples and more information.

2.4. Generalized Steinberg varieties. When analyzing the restriction of a Springer representation to parabolic subgroups of W , Springer introduced a generalization of \mathfrak{N} depending on a parabolic subgroup P and a nilpotent orbit in a Levi subgroup of P . Springer's ideas extend naturally to what we call "generalized Steinberg varieties." The results in this subsection may be found in [DR04].

Suppose \mathcal{P} is a conjugacy class of parabolic subgroups of G . The unipotent radical of a subgroup, P , in \mathcal{P} will be denoted by U_P . A G -equivariant function, c , from \mathcal{P} to the power set of \mathfrak{N} with the properties

- (1) $\mathfrak{u}_P \subseteq c(P) \subseteq \mathfrak{N} \cap \mathfrak{p}$ and
- (2) the image of $c(P)$ in $\mathfrak{p}/\mathfrak{u}_P$ is the closure of a single nilpotent adjoint P/U_P -orbit

is called a *Levi class function* on \mathcal{P} . Define

$$\tilde{\mathfrak{N}}_c^{\mathcal{P}} = \{ (x, P) \in \mathfrak{N} \times \mathcal{P} \mid x \in c(P) \}.$$

Let $\mu_c^{\mathcal{P}}: \tilde{\mathfrak{N}}_c^{\mathcal{P}} \rightarrow \mathfrak{N}$ denote the projection on the first factor. Notice that $\mu_c^{\mathcal{P}}$ is a proper morphism.

If \mathcal{Q} is another conjugacy class of parabolic subgroups of G and d is a Levi class function on \mathcal{Q} , then the *generalized Steinberg variety* determined by \mathcal{P} , \mathcal{Q} , c , and d is

$$X_{c,d}^{\mathcal{P},\mathcal{Q}} = \{ (x, P, Q) \in \mathfrak{N} \times \mathcal{P} \times \mathcal{Q} \mid x \in c(P) \cap d(Q) \} \cong \tilde{\mathfrak{N}}_c^{\mathcal{P}} \times_{\mathfrak{N}} \tilde{\mathfrak{N}}_d^{\mathcal{Q}}.$$

Since G acts on \mathfrak{N} , \mathcal{P} , and \mathcal{Q} , there is a diagonal action of G on $X_{c,d}^{\mathcal{P},\mathcal{Q}}$ for all \mathcal{P} , \mathcal{Q} , c , and d .

The varieties arising from this construction for some particular choices of \mathcal{P} , \mathcal{Q} , c , and d are worth noting.

- (1) When $\mathcal{P} = \mathcal{Q} = \mathcal{B}$, then $c(B') = d(B') = \{\mathfrak{u}_{B'}\}$ for every B' in \mathcal{B} , and so $X_{0,0}^{\mathcal{B},\mathcal{B}} = Z$ is the Steinberg variety of G .
- (2) In the special case when $c(P)$ and $d(Q)$ are as small as possible and correspond to the zero orbits in $\mathfrak{p}/\mathfrak{u}_P$ and $\mathfrak{q}/\mathfrak{u}_Q$ respectively: $c(P) = \mathfrak{u}_P$ and $d(Q) = \mathfrak{u}_Q$, we denote $X_{c,d}^{\mathcal{P},\mathcal{Q}}$ by $X_{0,0}^{\mathcal{P},\mathcal{Q}}$. We have $X_{0,0}^{\mathcal{P},\mathcal{Q}} \cong T^*\mathcal{P} \times_{\mathfrak{N}} T^*\mathcal{Q}$.
- (3) When $\mathcal{P} = \mathcal{Q} = \{G\}$, \mathfrak{D}_1 and \mathfrak{D}_2 are two nilpotent orbits in \mathfrak{g} , $c(G) = \overline{\mathfrak{D}_1}$ and $d(G) = \overline{\mathfrak{D}_2}$, then $X_{c,d}^{\{G\},\{G\}} \cong \overline{\mathfrak{D}_1} \cap \overline{\mathfrak{D}_2}$.

A special case that will arise frequently in the sequel is when $c(P)$ and $d(Q)$ are as large as possible and correspond to the regular, nilpotent orbits in $\mathfrak{p}/\mathfrak{u}_P$ and $\mathfrak{q}/\mathfrak{u}_Q$ respectively: $c(P) = \mathfrak{N} \cap \mathfrak{p}$ and $d(Q) = \mathfrak{N} \cap \mathfrak{q}$. We denote this generalized Steinberg variety simply by $X^{\mathcal{P},\mathcal{Q}}$.

Abusing notation slightly, we let $\mu: X_{c,d}^{\mathcal{P},\mathcal{Q}} \rightarrow \mathfrak{N}$ denote the projection on the first coordinate and $\pi: X_{c,d}^{\mathcal{P},\mathcal{Q}} \rightarrow \mathcal{P} \times \mathcal{Q}$ the projection on the second and third coordinates. We can then investigate the varieties $X_{c,d}^{\mathcal{P},\mathcal{Q}}$ using preimages of G -orbits in \mathfrak{N} and $\mathcal{P} \times \mathcal{Q}$ under μ and π as we did in §2.1 for the Steinberg variety. Special cases when at least one of $c(P)$ or $d(Q)$ is smooth turn out to be the most tractable. We will describe these cases in more detail below and refer the reader to [DR04] for more general results for arbitrary \mathcal{P} , \mathcal{Q} , c , and d .

Fix P in \mathcal{P} and Q in \mathcal{Q} with $B \subseteq P \cap Q$. Let W_P and W_Q denote the Weyl groups of (P, T) and (Q, T) respectively. We consider W_P and W_Q as subgroups of W .

For B' in \mathcal{B} , define $\pi_{\mathcal{P}}(B')$ to be the unique subgroup in \mathcal{P} containing B' . Then $\pi_{\mathcal{P}}: \mathcal{B} \rightarrow \mathcal{P}$ is a proper morphism with fibres isomorphic to P/B . Define

$$\eta: Z \rightarrow X^{\mathcal{P},\mathcal{Q}} \quad \text{by} \quad \eta(x, B', B'') = (x, \pi_{\mathcal{P}}(B'), \pi_{\mathcal{Q}}(B'')).$$

Then η depends on \mathcal{P} and \mathcal{Q} and is a proper, G -equivariant, surjective morphism.

Next, set $Z^{\mathcal{P},\mathcal{Q}} = \eta^{-1}(X_{0,0}^{\mathcal{P},\mathcal{Q}})$ and denote the restriction of η to $Z^{\mathcal{P},\mathcal{Q}}$ by η_1 . Then η_1 is also a proper, surjective, G -equivariant morphism. Moreover, the fibres of η_1 are all isomorphic to the smooth, complete variety $P/B \times Q/B$. More generally, define $Z_{c,d}^{\mathcal{P},\mathcal{Q}} = \eta^{-1}(X_{c,d}^{\mathcal{P},\mathcal{Q}})$.

The various varieties and morphisms we have defined fit together in a commutative diagram where the horizontal arrows are closed embeddings, the vertical arrows are proper maps, and the squares are cartesian:

$$\begin{array}{ccccc} Z^{\mathcal{P},\mathcal{Q}} & \longrightarrow & Z_{c,d}^{\mathcal{P},\mathcal{Q}} & \longrightarrow & Z \\ \eta_1 \downarrow & & \eta \downarrow & & \eta \downarrow \\ X_{0,0}^{\mathcal{P},\mathcal{Q}} & \longrightarrow & X_{c,d}^{\mathcal{P},\mathcal{Q}} & \longrightarrow & X^{\mathcal{P},\mathcal{Q}}. \end{array}$$

For w in W , define $Z_w^{\mathcal{P},\mathcal{Q}}$ to be the intersection $Z^{\mathcal{P},\mathcal{Q}} \cap Z_w$. Since $(0, B, wBw^{-1})$ is in $Z_w^{\mathcal{P},\mathcal{Q}}$ and η_1 is G -equivariant, it is straightforward to check that $Z_w^{\mathcal{P},\mathcal{Q}} \cong G \times^{B_w} (\mathfrak{u}_P \cap w\mathfrak{u}_Q)$. Thus $Z_w^{\mathcal{P},\mathcal{Q}}$ is smooth and irreducible.

The following statements are proved in [DR04].

- (1) For w in W , $\dim \eta(Z_w) \leq 2n$ with equality if and only if w has minimal length in $W_P w W_Q$. The variety $X^{\mathcal{P},\mathcal{Q}}$ is equidimensional with dimension equal to $2n$ and the set of irreducible components of $X^{\mathcal{P},\mathcal{Q}}$ is

$$\{ \eta(\overline{Z_w}) \mid w \text{ has minimal length in } W_P w W_Q \}.$$

- (2) For w in W , $Z_w^{\mathcal{P},\mathcal{Q}} = Z_w$ if and only if w has maximal length in $W_P w W_Q$. The variety $Z^{\mathcal{P},\mathcal{Q}}$ is equidimensional with dimension equal to $2n$ and the set of irreducible components of $Z^{\mathcal{P},\mathcal{Q}}$ is

$$\{ \overline{Z_w} \mid w \text{ has maximal length in } W_P w W_Q \}.$$

- (3) The variety $X_{0,0}^{\mathcal{P},\mathcal{Q}}$ is equidimensional with dimension equal to $\dim \mathfrak{u}_P + \dim \mathfrak{u}_Q$ and the set of irreducible components of $X_{0,0}^{\mathcal{P},\mathcal{Q}}$ is

$$\{ \eta_1(\overline{Z_w}) \mid w \text{ has maximal length in } W_P w W_Q \}.$$

- (4) For a Levi class function d on \mathcal{Q} , define ρ_d to be the number of irreducible components of $d(Q) \cap (\mathfrak{u} \cap \mathfrak{l}_Q)$, where L_Q is the Levi factor of Q that contains T . Then ρ_d is the number of orbital varieties for the open dense L_Q -orbit in $d(Q)/\mathfrak{u}_Q$ in the variety of nilpotent elements in $\mathfrak{q}/\mathfrak{u}_Q \cong \mathfrak{l}_Q$. The varieties $X_{0,d}^{\mathcal{B},\mathcal{Q}}$ are equidimensional with dimension $\frac{1}{2}(\dim \mathfrak{u} + \dim d(Q) + \dim \mathfrak{u}_Q)$ and $|W : W_{\mathcal{Q}}| \rho_d$ irreducible components.

Notice that the first statement relates minimal double coset representatives to regular orbits in Levi subalgebras and the third statement relates maximal double coset representatives to the zero orbits in Levi subalgebras.

The quantity ρ_d in the fourth statement is the degree of an irreducible representation of W_Q (see §3.5) and so $|W : W_{\mathcal{Q}}| \rho_d$ is the degree of an induced representation of W . The fact that $X_{0,d}^{\mathcal{B},\mathcal{Q}}$ has $|W : W_{\mathcal{Q}}| \rho_d$ irreducible components is numerical evidence for Conjecture 3.19 below.

3. HOMOLOGY

In this section we take up the rational Borel-Moore homology of the Steinberg variety and generalized Steinberg varieties. As mentioned in the Introduction, soon after Steinberg's original paper, Kazhdan and Lusztig [KL80] defined an action of $W \times W$ on the top Borel-Moore homology group of Z . They constructed this action by defining an action of the simple reflections in $W \times W$ on $H_i(Z)$ and showing that the defining relations of $W \times W$ are

satisfied. They then proved that the representation of $W \times W$ on $H_{4n}(Z)$ is equivalent to the two-sided regular representation of W , and following a suggestion of Springer, they gave a decomposition of $H_{4n}(Z)$ in terms of Springer representations of W . Springer representations of W will be described in §3.4–§3.6.

In the mid 1990s Ginzburg [CG97, Chapter 3] popularized a quite general convolution product construction that defines a \mathbb{Q} -algebra structure on $H_*(Z)$, the total Borel-Moore homology of Z , and a ring structure $K^{\overline{G}}(Z)$ (see the next section for $K^{\overline{G}}(Z)$). With this multiplication, $H_{4n}(Z)$ is a subalgebra isomorphic to the group algebra of W .

In this section, following [CG97, Chapter 3], [DR08b], and [HJ05] we will first describe the algebra structure of $H_*(Z)$, the decomposition of $H_{4n}(Z)$ in terms of Springer representations, and the $H_{4n}(Z)$ -module structure on $H_{2n}(Y)$ using elementary topological constructions. Then we will use a more sophisticated sheaf-theoretic approach to give an alternate description of $H_*(Z)$, a different version of the decomposition of $H_{4n}(Z)$ in terms of Springer representations, and to describe the Borel-Moore homology of some generalized Steinberg varieties.

3.1. Borel-Moore homology and convolution. We begin with a brief review of Borel-Moore homology, including the convolution and specialization constructions. The definitions and constructions in this subsection make sense in a very general setting, however for simplicity we will consider only complex algebraic varieties. More details and proofs may be found in [CG97, Chapter 2].

Suppose that X is a d -dimensional, quasi-projective, complex algebraic variety (not necessarily irreducible). Topological notions will refer to the Euclidean topology on X unless otherwise specified. Two exceptions to this convention are that we continue to denote the dimension of X as a complex variety by $\dim X$ and that “irreducible” means irreducible with respect to the Zariski topology. In particular, the topological dimension of X is $2 \dim X$.

Let $X \cup \{\infty\}$ be the one-point compactification of X . Then the i^{th} Borel-Moore homology space of X , denoted by $H_i(X)$, is defined by $H_i(X) = H_i^{\text{sing}}(X \cup \{\infty\}, \{\infty\})$, the relative, singular homology with rational coefficients of the pair $(X \cup \{\infty\}, \{\infty\})$. Define a graded \mathbb{Q} -vector space,

$$H_*(X) = \sum_{i \geq 0} H_i(X) - \text{the Borel-Moore homology of } X.$$

Borel-Moore homology is a bivariant theory in the sense of Fulton and MacPherson [FM81]: Suppose that $\phi: X \rightarrow Y$ is a morphism of varieties.

- If ϕ is proper, then there is an induced direct image map in Borel-Moore homology, $\phi_*: H_i(X) \rightarrow H_i(Y)$.
- If ϕ is smooth with f -dimensional fibres, then there is a pullback map in Borel-Moore homology, $\phi^*: H_i(Y) \rightarrow H_{i+2f}(X)$.

Moreover, if X is smooth and A and B are closed subvarieties of X , then there is an intersection pairing $\cap: H_i(A) \times H_j(B) \rightarrow H_{i+j-2d}(A \cap B)$. Although not reflected in the notation, this pairing depends on the triple (X, A, B) . In particular, the intersection pairing depends on the smooth ambient variety X .

In dimensions greater than or equal to $2 \dim X$, the Borel-Moore homology spaces of X are easily described. If $i > 2d$, then $H_i(X) = 0$, while the space $H_{2d}(X)$ has a natural

basis indexed by the d -dimensional irreducible components of X . If C is a d -dimensional irreducible component of X , then the homology class in $H_{2d}(X)$ determined by C is denoted by $[C]$.

For example, for the Steinberg variety, $H_i(Z) = 0$ for $i > 4n$ and the set $\{ [\overline{Z_w}] \mid w \in W \}$ is a basis of $H_{4n}(Z)$. Similarly, for the conormal variety, $H_i(Y) = 0$ for $i > 2n$ and the set $\{ [\overline{Y_w}] \mid w \in W \}$ is a basis of $H_{2n}(Y)$.

Suppose that for $i = 1, 2, 3$, M_i is a smooth, connected, d_i -dimensional variety. For $1 \leq i < j \leq 3$, let $p_{i,j}: M_1 \times M_2 \times M_3 \rightarrow M_i \times M_j$ denote the projection. Notice that each $p_{i,j}$ is smooth and so the pullback maps $p_{i,j}^*$ in Borel-Moore homology are defined.

Now suppose $Z_{1,2}$ is a closed subset of $M_1 \times M_2$ and $Z_{2,3}$ is a closed subvariety of $M_2 \times M_3$. Define $Z_{1,3} = Z_{1,2} \circ Z_{2,3}$ to be the composition of the relations $Z_{1,2}$ and $Z_{2,3}$. Then

$$Z_{1,3} = \{ (m_1, m_3) \in M_1 \times M_3 \mid \exists m_2 \in M_2 \text{ with } (m_1, m_2) \in Z_{1,2} \text{ and } (m_2, m_3) \in Z_{2,3} \}.$$

In order to define the convolution product, we assume in addition that the restriction

$$p_{1,3}: p_{1,2}^{-1}(Z_{1,2}) \cap p_{2,3}^{-1}(Z_{2,3}) \rightarrow Z_{1,3}$$

is a proper morphism. Thus, there is a direct image map

$$(p_{1,3})_*: H_i(p_{1,2}^{-1}(Z_{1,2}) \cap p_{2,3}^{-1}(Z_{2,3})) \rightarrow H_i(Z_{1,3})$$

in Borel-Moore homology. The *convolution product*, $H_i(Z_{1,2}) \times H_j(Z_{2,3}) \xrightarrow{*} H_{i+j-2d_2}(Z_{1,3})$ is then defined by

$$c * d = (p_{1,3})_* (p_{1,2}^*(c) \cap p_{2,3}^*(d))$$

where \cap is the intersection pairing determined by the subsets $Z_{1,2} \times M_3$ and $M_1 \times Z_{2,3}$ of $M_1 \times M_2 \times M_3$. It is a straightforward exercise to show that the convolution product is associative.

The convolution construction is particularly well adapted to fibred products. Fix a “base” variety, N , which is not necessarily smooth, and suppose that for $i = 1, 2, 3$, $f_i: M_i \rightarrow N$ is a proper morphism. Then taking $Z_{1,2} = M_1 \times_N M_2$, $Z_{2,3} = M_2 \times_N M_3$, and $Z_{1,3} = M_1 \times_N M_3$, we have a convolution product $H_i(M_1 \times_N M_2) \times H_j(M_2 \times_N M_3) \xrightarrow{*} H_{i+j-2d_2}(M_1 \times_N M_3)$.

As a special case, when $M_1 = M_2 = M_3 = M$ and $f_1 = f_2 = f_3 = f$, then taking $Z_{i,j} = M \times_N M$ for $1 \leq i < j \leq 3$, the convolution product defines a multiplication on $H_*(M \times_N M)$ so that $H_*(M \times_N M)$ is a \mathbb{Q} -algebra with identity. The identity in $H_*(M \times_N M)$ is $[M_\Delta]$ where M_Δ is the diagonal in $M \times M$. If $d = \dim M$, then $H_i(M \times_N M) * H_j(M \times_N M) \subseteq H_{i+j-2d}(M \times_N M)$ and so $H_{2d}(M \times_N M)$ is a subalgebra and $\oplus_{i < 2d} H_i(M \times_N M)$ is a nilpotent, two-sided ideal.

Another special case is when M and M' are smooth and $f: M \rightarrow N$ and $f': M' \rightarrow N$ are proper maps. Then taking $Z_{1,2} = M \times_N M$ and $Z_{2,3} = M \times_N M'$, the convolution product defines a left $H_*(M \times_N M)$ -module structure on $H_*(M \times_N M')$. A further special case of this construction is when $M' = A$ is a smooth, closed subset of N and $f': A \rightarrow N$ is the inclusion. Then $M \times_N A \cong f^{-1}A$ and the convolution product defines a left $H_*(M \times_N M)$ -module structure on $H_*(f^{-1}(A))$. This construction will be exploited extensively in §3.5.

As an example, recall that $Z \cong \tilde{\mathfrak{N}} \times_{\mathfrak{N}} \tilde{\mathfrak{N}}$ where $\mu: \tilde{\mathfrak{N}} \rightarrow \mathfrak{N}$ is a proper map. Applying the constructions in the last two paragraphs to Z and to M' , where $M' = Y = \mu^{-1}(\mathfrak{u})$ and $M' = \mathcal{B}_x = \mu^{-1}(x)$ for x in \mathfrak{N} , we obtain the following proposition.

Proposition 3.1. *The convolution product defines a \mathbb{Q} -algebra structure on $H_*(Z)$ so that $H_{4n}(Z)$ is a $|W|$ -dimensional subalgebra and $\bigoplus_{i < 4n} H_i(Z)$ is a two-sided, nilpotent ideal. Moreover, the convolution product defines left $H_*(Z)$ -module structures on $H_*(Y)$ and on $H_*(\mathcal{B}_x)$ for x in \mathfrak{N} .*

In the next two subsections we will make use of the following specialization construction in Borel-Moore homology due to Fulton and MacPherson [FM81, §3.4].

Suppose that our base variety N is smooth and s -dimensional. Fix a distinguished point n_0 in N and set $N^* = N \setminus \{n_0\}$. Let M be a variety, not necessarily smooth, and suppose that $\phi: M \rightarrow N$ is a surjective morphism. Set $M_0 = \phi^{-1}(n_0)$ and $M^* = \phi^{-1}(N^*)$. Assume that the restriction $\phi|_{M^*}: M^* \rightarrow N^*$ is a locally trivial fibration. Then there is a “specialization” map in Borel-Moore homology, $\lim: H_i(M^*) \rightarrow H_{i-2s}(M_0)$ (see [CG97, §2.6]). It is shown in [CG97, §2.7] that when all the various constructions are defined, specialization commutes with convolution: $\lim(c * d) = \lim c * \lim d$.

3.2. The specialization construction and $H_{4n}(Z)$. Chriss and Ginzburg [CG97, §3.4] use the specialization construction to show that $H_{4n}(Z)$ is isomorphic to the group algebra $\mathbb{Q}[W]$. We present their construction in this subsection. In the next subsection we show that the specialization construction can also be used to show that $H_*(Z)$ is isomorphic to the smash product of the group algebra of W and the coinvariant algebra of W .

We would like to apply the specialization construction when the variety M_0 is equal Z . In order to do this, we need varieties that are larger than \mathfrak{N} , $\tilde{\mathfrak{N}}$, and Z .

Define

$$\tilde{\mathfrak{g}} = \{(x, B') \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}'\} \quad \text{and} \quad \hat{Z} = \{(x, B', B'') \in \mathfrak{g} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{b}' \cap \mathfrak{b}''\}.$$

Abusing notation again, let $\mu: \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ and $\mu_z: \hat{Z} \rightarrow \mathfrak{g}$ denote the projections on the first factors and let $\pi: \hat{Z} \rightarrow \mathcal{B} \times \mathcal{B}$ denote the projection on the second and third factors.

For w in W define $\hat{Z}_w = \pi^{-1}(G(B, wBw^{-1}))$. Then $\hat{Z}_w \cong G \times^{B_w} \mathfrak{b}_w$. Therefore, $\dim \hat{Z}_w = \dim \mathfrak{g}$ and the closures of the \hat{Z}_w 's for w in W are the irreducible components of \hat{Z} .

As with Z , we have an alternate description of \hat{Z} as $(\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}) \times_{\mathfrak{g} \times \mathfrak{g}} \mathfrak{g}$. However, in contrast to the situation in §2.3, where $Z \cong \{(x, -x, B', B'') \in \mathfrak{N} \times \mathfrak{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{b}' \cap \mathfrak{b}'' \cap \mathfrak{N}\}$, in this section we use that $\hat{Z} \cong \{(x, B', x, B'') \in \mathfrak{g} \times \mathcal{B} \times \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}' \cap \mathfrak{b}''\}$. In particular, we will frequently identify \hat{Z} with the subvariety of $\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$ consisting of all pairs $((x, B'), (x, B''))$ with x in $\mathfrak{b}' \cap \mathfrak{b}''$. Similarly, we will frequently identify Z with the subvariety of $\mathfrak{N} \times \mathfrak{N}$ consisting of all pairs $((x, B'), (x, B''))$ with x in $\mathfrak{N} \cap \mathfrak{b}' \cap \mathfrak{b}''$.

For (x, gBg^{-1}) in $\tilde{\mathfrak{g}}$, define $\nu(x, gBg^{-1})$ to be the projection of $g^{-1} \cdot x$ in \mathfrak{t} . Then $\nu: \tilde{\mathfrak{g}} \rightarrow \mathfrak{t}$ is a surjective morphism. For w in W , let $\Gamma_{w^{-1}} = \{(h, w^{-1} \cdot h) \mid h \in \mathfrak{t}\} \subseteq \mathfrak{t} \times \mathfrak{t}$ denote the graph of the action of w^{-1} on \mathfrak{t} and define

$$\Lambda_w = \hat{Z} \cap (\nu \times \nu)^{-1}(\Gamma_{w^{-1}}) = \{(x, B', B'') \in \hat{Z} \mid \nu(x, B'') = w^{-1}\nu(x, B')\}.$$

The spaces we have defined so far fit into a commutative diagram with cartesian squares where $\delta: \mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}$ is the diagonal map:

$$(3.2) \quad \begin{array}{ccccc} \Lambda_w & \xrightarrow{\quad} & \widehat{Z} & \xrightarrow{\mu_z} & \mathfrak{g} \\ \downarrow & & \downarrow & & \downarrow \delta \\ (\nu \times \nu)^{-1}(\Gamma_{w^{-1}}) & \xrightarrow{\quad} & \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}} & \xrightarrow{\mu \times \mu} & \mathfrak{g} \times \mathfrak{g} \\ \downarrow & & \downarrow \nu \times \nu & & \\ \Gamma_{w^{-1}} & \xrightarrow{\quad} & \mathfrak{t} \times \mathfrak{t} & & \end{array}$$

Let $\nu_w: \Lambda_w \rightarrow \Gamma_{w^{-1}}$ denote the composition of the leftmost vertical maps in (3.2), so ν_w is the restriction of $\nu \times \nu$ to Λ_w . We will consider subsets of \widehat{Z} of the form $\nu_w^{-1}(S')$ for $S' \subseteq \Gamma_{w^{-1}}$. Thus, for h in \mathfrak{t} we define $\Lambda_w^h = \nu_w^{-1}(h, w^{-1}h)$. Notice in particular that $\Lambda_w^0 = Z$. More generally, for a subset S of \mathfrak{t} we define $\Lambda_w^S = \coprod_{h \in S} \Lambda_w^h$. Then $\Lambda_w^S = \nu_w^{-1}(S')$, where S' is the graph of w^{-1} restricted to S .

Let $\mathfrak{t}_{\text{reg}}$ denote the set of regular elements in \mathfrak{t} . For w in W , define $\widetilde{w}: G/T \times \mathfrak{t}_{\text{reg}} \rightarrow G/T \times \mathfrak{t}_{\text{reg}}$ by $\widetilde{w}(gT, h) = (gwT, w^{-1}h)$. The rule $(gT, h) \mapsto (g \cdot h, gB)$ defines an isomorphism of varieties $f: G/T \times \mathfrak{t}_{\text{reg}} \xrightarrow{\cong} \widetilde{\mathfrak{g}}_{\text{rs}}$, where $\widetilde{\mathfrak{g}}_{\text{rs}} = \mu^{-1}(G \cdot \mathfrak{t}_{\text{reg}})$. We denote the automorphism $f \circ \widetilde{w} \circ f^{-1}$ of $\widetilde{\mathfrak{g}}_{\text{rs}}$ also by \widetilde{w} .

We now have all the notation in place for the specialization construction. Fix an element w in W and a one-dimensional subspace, ℓ , of \mathfrak{t} so that $\ell \cap \mathfrak{t}_{\text{reg}} = \ell \setminus \{0\}$. The line ℓ is our base space and the distinguished point in ℓ is 0. As above, we set $\ell^* = \ell \setminus \{0\}$. We denote the restriction of ν_w to Λ_w^ℓ again by ν_w . Then $\nu_w: \Lambda_w^\ell \rightarrow \ell$ is a surjective morphism with $\nu_w^{-1}(0) = Z$ and $\nu_w^{-1}(\ell^*) = \Lambda_w^{\ell^*}$. We will see below that the restriction $\Lambda_w^{\ell^*} \rightarrow \ell^*$ is a locally trivial fibration and so a specialization map

$$(3.3) \quad \lim: H_{i+2}(\Lambda_w^{\ell^*}) \rightarrow H_i(Z)$$

is defined.

It is not hard to check that the variety $\Lambda_w^{\ell^*}$ is the graph of $\widetilde{w}|_{\widetilde{\mathfrak{g}}^{\ell^*}}: \widetilde{\mathfrak{g}}^{\ell^*} \rightarrow \widetilde{\mathfrak{g}}^{w^{-1}(\ell^*)}$, where for an arbitrary subset S of \mathfrak{t} , $\widetilde{\mathfrak{g}}^S$ is defined to be $\nu^{-1}(S) = \{(x, B') \in \widetilde{\mathfrak{g}} \mid \nu(x, B') \in S\}$. It follows that for h in ℓ^* we have $\nu_w^{-1}(h) = \Lambda_w^h \cong G/T$ and that $\Lambda_w^{\ell^*} \rightarrow \ell^*$ is a locally trivial fibration. Moreover, $\Lambda_w^{\ell^*} \cong \widetilde{\mathfrak{g}}^{\ell^*}$ and hence is an irreducible, $(2n+1)$ -dimensional variety. Therefore, $H_{4n+2}(\Lambda_w^{\ell^*})$ is one-dimensional with basis $\{[\Lambda_w^{\ell^*}]\}$. Taking $i = 4n$ in (3.3), we define

$$\lambda_w = \lim([\Lambda_w^{\ell^*}])$$

in $H_{4n}(Z)$.

Because $\Lambda_w^{\ell^*}$ is a graph, it follows easily from the definitions that for y in W , there is a convolution product

$$H_*(\Lambda_w^{\ell^*}) \times H_*(\Lambda_y^{w^{-1}\ell^*}) \xrightarrow{*} H_*(\Lambda_{wy}^{\ell^*})$$

and that $[\Lambda_w^{\ell^*}] * [\Lambda_y^{w^{-1}\ell^*}] = [\Lambda_{wy}^{\ell^*}]$. Because specialization commutes with convolution, we have $\lambda_w * \lambda_y = \lambda_{wy}$ for all w and y in W .

Chriss and Ginzburg [CG97, §3.4] have proved the following:

- (1) The element λ_w in $H_{4n}(Z)$ does not depend on the choice of ℓ .

- (2) The expansion of λ_w as a linear combination of the basis elements $[\overline{Z_y}]$ of $H_{4n}(Z)$ has the form $\lambda_w = [\overline{Z_w}] + \sum_{y < w} a_{y,w} [\overline{Z_y}]$ where \leq is the Bruhat order on W .

These results prove the following theorem.

Theorem 3.4. *With the notation as above, the assignment $w \mapsto \lambda_w$ extends to an algebra isomorphism $\mathbb{Q}[W] \xrightarrow{\cong} H_{4n}(Z)$.*

3.3. The Borel-Moore homology of Z and coinvariants. Now consider

$$Z_1 = \{ (x, B', B') \in \mathfrak{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{b}' \}.$$

Then Z_1 may be identified with the diagonal in $\tilde{\mathfrak{N}} \times \tilde{\mathfrak{N}}$. It follows that Z_1 is closed in Z and isomorphic to $\tilde{\mathfrak{N}}$.

Since $\tilde{\mathfrak{N}} \cong T^*\mathcal{B}$, it follows from the Thom isomorphism in Borel-Moore homology that $H_{i+2n}(Z_1) \cong H_i(\mathcal{B})$ for all i . Since \mathcal{B} is smooth and compact, $H_i(\mathcal{B}) \cong H^{2n-i}(\mathcal{B})$ by Poincaré duality. Therefore, $H_{4n-i}(Z_1) \cong H^i(\mathcal{B})$ for all i .

The cohomology of \mathcal{B} is well-understood: there is an isomorphism of graded algebras, $H^*(\mathcal{B}) \cong \text{Coinv}_*(W)$ where $\text{Coinv}_*(W)$ is the coinvariant algebra of W with generators in degree 2. It follows that $H_j(Z_1) = 0$ if j is odd, $H_{4n-2i}(Z_1) \cong \text{Coinv}_{2i}(W)$ for $0 \leq i \leq n$.

The following is proved in [DR08b].

- (1) There is a convolution product on $H_*(Z_1)$. With this product, $H_*(Z_1)$ is a commutative \mathbb{Q} -algebra and there is an isomorphism of graded \mathbb{Q} -algebras

$$\beta: \text{Coinv}_*(W) \xrightarrow{\cong} H_{4n-*}(Z_1).$$

- (2) If $\iota: Z_1 \rightarrow Z$ denotes the inclusion, then the direct image map in Borel-Moore homology, $\iota_*: H_*(Z_1) \rightarrow H_*(Z)$, is an injective ring homomorphism.
- (3) If we identify $H_*(Z_1)$ with its image in $H_*(Z)$ as in (b), then the linear transformation given by the convolution product

$$H_i(Z_1) \otimes H_{4n}(Z) \xrightarrow{*} H_i(Z)$$

is an isomorphism of vector spaces for $0 \leq i \leq 4n$.

The algebra $\text{Coinv}_*(W)$ has a natural action of W by algebra automorphisms and the isomorphism β in (a) is in fact an isomorphism of W -algebras. The W -algebra structure on $H_*(Z_1)$ is described as follows.

Fix w in W and identify $H_*(Z_1)$ with its image in $H_*(Z)$. Then

$$\lambda_w * H_i(Z_1) * \lambda_{w^{-1}} = H_i(Z_1).$$

Therefore, conjugation by λ_w defines a W -algebra structure on $H_*(Z_1)$. With this W -algebra structure, the isomorphism $\beta: \text{Coinv}_*(W) \xrightarrow{\cong} H_{4n-*}(Z_1)$ in (a) is an isomorphism of W -algebras.

Using the natural action of W on $\text{Coinv}(W)$, we can define the smash product algebra $\text{Coinv}(W) \rtimes \mathbb{Q}[W]$. We suppose that $\text{Coinv}(W) \rtimes \mathbb{Q}[W]$ is graded by $(\text{Coinv}(W) \rtimes \mathbb{Q}[W])_i = \text{Coinv}_i(W) \otimes \mathbb{Q}[W]$. Then combining Theorem 3.4, item (3) above, and the fact that β is an isomorphism of W -algebras, we obtain the following theorem giving an explicit description of the structure of $H_*(Z)$.

Theorem 3.5. *The composition*

$$\mathrm{Coinv}_*(W) \rtimes \mathbb{Q}[W] \xrightarrow{\beta \otimes \alpha} H_{4n-*}(Z_1) \otimes H_{4n}(Z) \xrightarrow{*} H_{4n-*}(Z)$$

is an isomorphism of graded \mathbb{Q} -algebras.

3.4. Springer representations of W . Springer [Spr76] [Spr78] has given a case-free construction of the irreducible representations of W . He achieves this by defining an action of W on $H^*(\mathcal{B}_x)$ for x in \mathfrak{N} . Define $d_x = \dim \mathcal{B}_x$ and let $C(x) = Z_G(x)/Z_G^0(x)$. Then the centralizer in G of x acts on \mathcal{B}_x and so $C(x)$ acts on $H^*(\mathcal{B}_x)$. Springer shows that if ϕ is an irreducible representation of $C(x)$ and $H^{2d_x}(\mathcal{B}_x)_\phi$ is the homogeneous component of $H^{2d_x}(\mathcal{B}_x)$ corresponding to ϕ , then $H^{2d_x}(\mathcal{B}_x)_\phi$ is W -stable and is either zero or affords an irreducible representation of W . He shows furthermore that every irreducible representation of W arises in this way.

We have seen in §3.1 that for x in \mathfrak{N} , the convolution product defines a left $H_{4n}(Z)$ -module structure on $H_*(\mathcal{B}_x)$ and in §3.2 that $H_{4n}(Z) \cong \mathbb{Q}[W]$. Thus, we obtain a representation of W on $H_*(\mathcal{B}_x)$. Because \mathcal{B}_x is projective, and hence compact, $H^*(\mathcal{B}_x)$ is the linear dual of $H_*(\mathcal{B}_x)$ and so we obtain a representation of W on $H^*(\mathcal{B}_x)$.

In the next subsection we use topological techniques to decompose the two-sided regular representation of $H_{4n}(Z)$ into irreducible sub-bimodules and describe these sub-bimodules explicitly in terms of the irreducible $H_{4n}(Z)$ -submodules of $H_{2d_x}(\mathcal{B}_x)$ for x in \mathfrak{N} . In §3.6 we use sheaf theoretic techniques to decompose the representation of $\mathbb{Q}[W] \cong H_{4n}(Z)$ on $H_*(\mathcal{B}_x)$ into irreducible constituents.

As above, the component group $C(x)$ acts on $H_*(\mathcal{B}_x)$. It is easy to check that the $C(x)$ -action and the $H_{4n}(Z)$ -action commute. Therefore, up to isomorphism, the representation of W on $H_*(\mathcal{B}_x)$ depends only on the G -orbit of x and the isotopic components for the $C(x)$ -action afford representations of W .

It follows from results of Hotta [Hot82] that the representations of W on $H_*(\mathcal{B}_x)$ constructed using the convolution product and the isomorphism $\mathbb{Q}[W] \cong H_{4n}(Z)$ are the same as the representations originally constructed by Springer tensored with the sign representation of W .

As an example, consider the special case corresponding to the trivial representation of $C(x)$: $H_{2d_x}(\mathcal{B}_x)^{C(x)}$, the $C(x)$ -invariants in $H_{2d_x}(\mathcal{B}_x)$. Let \mathcal{Irr}_x denote the set of irreducible components of \mathcal{B}_x . Then $\{[C] \mid C \in \mathcal{Irr}_x\}$ is a basis of $H_{2d_x}(\mathcal{B}_x)$. The group $C(x)$ acts on $H_{2d_x}(\mathcal{B}_x)$ by permuting this basis: $g[C] = [gC]$ for g in $Z_G(x)$ and C in \mathcal{Irr}_x . Thus, the orbit sums index a basis of $H_{2d_x}(\mathcal{B}_x)^{C(x)}$. We have seen in §2.2 that there is a bijection between the orbits of $C(x)$ on \mathcal{Irr}_x and the set of orbital varieties for \mathfrak{D} where \mathfrak{D} is the G -orbit of x . Thus, $H_{2d_x}(\mathcal{B}_x)^{C(x)}$ affords a representation of W and has a basis naturally indexed by the set of orbital varieties for \mathfrak{D} . It follows from the general results stated above and discussed in more detail in the following two subsections that this representation is irreducible.

3.5. More on the top Borel-Moore homology of Z . We saw in Theorem 3.4 that $H_{4n}(Z) \cong \mathbb{Q}[W]$. In this subsection we follow the argument in [CG97, §3.5]. First we obtain a filtration of $H_{4n}(Z)$ by two sided ideals indexed by the set of nilpotent orbits in \mathfrak{N} and then we describe the decomposition of the associated graded ring into minimal two-sided ideals. In particular, we obtain a case-free construction and parametrization of the irreducible representations of W . As explained in the introduction, a very similar result was

first obtained using different methods by Kazhdan and Lusztig [KL80], following an idea of Springer.

Recall that orbit closure defines a partial order on the set of nilpotent orbits in \mathfrak{N} : $\mathfrak{O}_1 \leq \mathfrak{O}_2$ if $\mathfrak{O}_1 \subseteq \overline{\mathfrak{O}_2}$. For a nilpotent orbit, \mathfrak{O} , define $\partial\overline{\mathfrak{O}} = \overline{\mathfrak{O}} \setminus \mathfrak{O} = \{\mathfrak{O}' \mid \mathfrak{O}' < \mathfrak{O}\}$ and set $Z_{\overline{\mathfrak{O}}} = \mu_z^{-1}(\overline{\mathfrak{O}})$, and $Z_{\partial\overline{\mathfrak{O}}} = \mu_z^{-1}(\partial\overline{\mathfrak{O}})$. Notice that $\partial\overline{\mathfrak{O}}$ is a closed subvariety of \mathfrak{N} . Define $W_{\overline{\mathfrak{O}}} = \cup_{\mathfrak{O} \subseteq \overline{\mathfrak{O}}} W_{\mathfrak{O}}$ and $W_{\partial\overline{\mathfrak{O}}} = \cup_{\mathfrak{O} \subseteq \partial\overline{\mathfrak{O}}} W_{\mathfrak{O}}$, where the union is taken over the nilpotent orbits contained in $\overline{\mathfrak{O}}$ and $\partial\overline{\mathfrak{O}}$ respectively.

It follows from the results in §2.1 and §3.1 that $\{\overline{[Z_w]} \mid w \in W_{\overline{\mathfrak{O}}}\}$ is a basis of $H_{4n}(Z_{\overline{\mathfrak{O}}})$.

If we take $f_i: M_i \rightarrow N$ to be $\mu: \tilde{\mathfrak{N}} \rightarrow \mathfrak{N}$ for $i = 1, 2, 3$ and $Z_{i,j} = Z_{\overline{\mathfrak{O}}}$ for $1 \leq i \neq j \leq 3$, then the convolution product construction in §3.1 defines the structure of a \mathbb{Q} -algebra on $H_*(Z_{\overline{\mathfrak{O}}})$ and $H_{4n}(Z_{\overline{\mathfrak{O}}})$ is a subalgebra. Similarly, taking $Z_{1,2} = Z$ and $Z_{2,3} = Z_{1,3} = Z_{\overline{\mathfrak{O}}}$, the convolution product defines a left $H_*(Z)$ -module structure on $H_*(Z_{\overline{\mathfrak{O}}})$ that is compatible with the algebra structure on $H_*(Z_{\overline{\mathfrak{O}}})$ in the sense that $a * (y * z) = (a * y) * z$ for a in $H_*(Z)$ and y and z in $H_*(\overline{\mathfrak{O}})$. Taking $Z_{1,2} = Z_{1,3} = Z_{\overline{\mathfrak{O}}}$ and $Z_{1,2} = Z$, we get a right $H_*(Z)$ -module structure on $H_*(Z_{\overline{\mathfrak{O}}})$ that commutes with the left $H_*(Z)$ -module structure and is compatible with the algebra structure. Thus, we see that $H_{4n}(Z_{\overline{\mathfrak{O}}})$ is a $|W_{\overline{\mathfrak{O}}}|$ -dimensional algebra with a compatible $H_{4n}(Z)$ -bimodule structure.

Arguing as in the last two paragraphs with $Z_{\overline{\mathfrak{O}}}$ replaced by $Z_{\partial\overline{\mathfrak{O}}}$, we see that $H_{4n}(Z_{\partial\overline{\mathfrak{O}}})$ is a $|W_{\partial\overline{\mathfrak{O}}}|$ -dimensional algebra with a compatible $H_{4n}(Z)$ -bimodule structure.

The inclusions $Z_{\partial\overline{\mathfrak{O}}} \subseteq Z_{\overline{\mathfrak{O}}} \subseteq Z$ induce injective, $H_{4n}(Z) \times H_{4n}(Z)$ -linear ring homomorphisms, $H_{4n}(Z_{\partial\overline{\mathfrak{O}}}) \rightarrow H_{4n}(Z_{\overline{\mathfrak{O}}}) \rightarrow H_{4n}(Z)$, and so we may identify $H_{4n}(Z_{\partial\overline{\mathfrak{O}}})$ and $H_{4n}(Z_{\overline{\mathfrak{O}}})$ with their images in $H_{4n}(Z)$ and consider $H_{4n}(Z_{\partial\overline{\mathfrak{O}}})$ and $H_{4n}(Z_{\overline{\mathfrak{O}}})$ as two-sided ideals in $H_{4n}(Z)$.

The two-sided ideals $H_{4n}(Z_{\overline{\mathfrak{O}}})$ define a filtration of $H_{4n}(Z)$ indexed by the set of nilpotent orbits. Thus, to describe the decomposition of the associated graded algebra into minimal two-sided ideals, we need to analyze the quotients $H_{4n}(Z_{\overline{\mathfrak{O}}})/H_{4n}(Z_{\partial\overline{\mathfrak{O}}})$. Because $H_{4n}(Z)$ is semisimple (it is isomorphic to $\mathbb{Q}[W]$), this will also describe the two-sided regular representation of $H_{4n}(Z)$ into minimal sub-bimodules and give a case-free construction of the irreducible representations of W .

For a G -orbit, \mathfrak{O} , define $H_{\mathfrak{O}}$ to be the quotient $H_{4n}(Z_{\overline{\mathfrak{O}}})/H_{4n}(Z_{\partial\overline{\mathfrak{O}}})$. Then $\dim H_{\mathfrak{O}} = |W_{\mathfrak{O}}|$ and $H_{\mathfrak{O}}$ is an $H_{4n}(Z)$ -bimodule with a compatible \mathbb{Q} -algebra structure inherited from the convolution product on $H_{4n}(Z)$.

Now fix a G -orbit \mathfrak{O} and an element x is in \mathfrak{O} . Set $Z_x = \mu_z^{-1}(x)$. Then clearly $Z_x \cong \mathcal{B}_x \times \mathcal{B}_x$ and $\dim Z_x = 2d_x$. The centralizer of x acts diagonally on Z_x , and so the component group, $C(x)$, acts on $H_*(Z_x)$. Thus, $H_{4d_x}(Z_x)^{C(x)} \cong H_{4d_x}(\mathcal{B}_x \times \mathcal{B}_x)^{C(x)}$ has a basis indexed by the $C(x)$ -orbits on the set of irreducible components of $\mathcal{B}_x \times \mathcal{B}_x$. We saw in §2.1 that there is a bijection between the $C(x)$ -orbits on the set of irreducible components of $\mathcal{B}_x \times \mathcal{B}_x$ and the two-sided Steinberg cell corresponding to \mathfrak{O} . Therefore, the dimension of $H_{4d_x}(Z_x)^{C(x)}$ is $|W_{\mathfrak{O}}| = \dim H_{\mathfrak{O}}$.

As for $Z_{\overline{\mathfrak{O}}}$ and $Z_{\partial\overline{\mathfrak{O}}}$, if we take $f_i: M_i \rightarrow N$ to be $\mu: \tilde{\mathfrak{N}} \rightarrow \mathfrak{N}$ for $i = 1, 2, 3$, then for suitable choices of $Z_{i,j}$ for $1 \leq i < j \leq 3$, the convolution product defines a \mathbb{Q} -algebra structure and a compatible $H_*(Z)$ -bimodule structure on $H_{4d_x}(Z_x)$. It is straightforward to check that $H_{4d_x}(Z_x)^{C(x)}$ is a subalgebra and an $H_*(Z)$ -sub-bimodule of $H_{4d_x}(Z_x)$.

The group $C(x)$ acts diagonally on $H_{2d_x}(\mathcal{B}_x) \otimes H_{2d_x}(\mathcal{B}_x)$ and it follows from the Künneth formula that

$$(3.6) \quad H_{4d_x}(Z_x)^{C(x)} \cong (H_{2d_x}(\mathcal{B}_x) \otimes H_{2d_x}(\mathcal{B}_x))^{C(x)}.$$

The convolution product defines left and right $H_*(Z)$ -module structures on $H_*(\mathcal{B}_x)$ and the isomorphism in (3.6) is as $H_*(Z)$ -bimodules, where $H_*(Z)$ acts on the right-hand side by acting on the left on the first $H_{2d_x}(\mathcal{B}_x)$ and on the right on the second $H_{2d_x}(\mathcal{B}_x)$.

Fix a set, \mathfrak{S} , of G -orbit representatives in \mathfrak{N} . The next proposition has been proved by Kazhdan and Lusztig [KL80] and Chriss and Ginzburg [CG97, §3.5]. An alternate argument has also been given by Hinich and Joseph [HJ05, §4].

Proposition 3.7. *There is an algebra isomorphism $H_{\mathfrak{D}} \cong H_{4n}(Z)^{C(x)}$ and $H_{4n}(Z)$ -bimodule isomorphisms*

$$H_{\mathfrak{D}} \cong H_{4n}(Z)^{C(x)} \cong (H_{2d_x}(\mathcal{B}_x) \otimes H_{2d_x}(\mathcal{B}_x))^{C(x)}.$$

For $\mathfrak{D} = \{0\}$, the $H_{4n}(Z)$ -bimodule $H_{\mathfrak{D}}$ corresponds to the trivial representation of W under the isomorphism $H_{4n}(Z) \cong \mathbb{Q}[W]$. For \mathfrak{D} the regular nilpotent orbit, the $H_{4n}(Z)$ -bimodule $H_{\mathfrak{D}}$ corresponds to the sign representation of W . In general however, $H_{\mathfrak{D}}$ is not a minimal two-sided ideal in the associated graded ring, $\text{gr } H_{4n}(Z)$, and not an irreducible $H_{4n}(Z)$ -bimodule. To obtain the decomposition of $\text{gr } H_{4n}(Z)$ into irreducible $H_{4n}(Z)$ -bimodules, we need to decompose each $H_{2d_x}(\mathcal{B}_x)$ into $C(x)$ -isotopic components.

For an irreducible representation of $C(x)$ with character ϕ , denote the ϕ -isotopic component of $C(x)$ on $H_{2d_x}(\mathcal{B}_x)$ by $H_{2d_x}(\mathcal{B}_x)_{\phi}$. Define $\widehat{C(x)}$ to be the set of ϕ with $H_{2d_x}(\mathcal{B}_x)_{\phi} \neq 0$. We saw in the last subsection that the trivial character of $C(x)$ is always an element of $\widehat{C(x)}$. The sets $\widehat{C(x)}$ have been computed explicitly in all cases, see [Car85, §13.3]. For example, if $G = \text{GL}_n(\mathbb{C})$, then $G_C(x)$ is connected and so $C(x) = 1$ for all x in \mathfrak{N} , and so $\widehat{C(x)}$ contains all irreducible characters of $C(x)$. In general $\widehat{C(x)}$ does not contain all irreducible characters of $C(x)$.

Recall from §3.4 that for each ϕ , $H_{2d_x}(\mathcal{B}_x)_{\phi}$ is an $H_{4n}(Z)$ -submodule of $H_{2d_x}(\mathcal{B}_x)$.

The next theorem is proved directly in [KL80] and [CG97, §3.5]. It also follows from the sheaf-theoretic approach to Borel-Moore homology described below.

Theorem 3.8. *There is an isomorphism of $H_{4n}(Z)$ -bimodules,*

$$(H_{2d_x}(\mathcal{B}_x) \otimes H_{2d_x}(\mathcal{B}_x))^{C(x)} \cong \bigoplus_{\phi \in \widehat{C(x)}} \text{End}_{\mathbb{Q}}(H_{2d_x}(\mathcal{B}_x)_{\phi}).$$

Moreover, $H_{2d_x}(\mathcal{B}_x)_{\phi}$ is a simple $\text{gr } H_{4n}(Z)$ -module for every ϕ in $\widehat{C(x)}$ and the decomposition

$$\text{gr } H_{4n}(Z) \cong \bigoplus_{x \in \mathfrak{S}} \bigoplus_{\phi \in \widehat{C(x)}} \text{End}_{\mathbb{Q}}(H_{2d_x}(\mathcal{B}_x)_{\phi})$$

is a decomposition of $H_{4n}(Z)$ into minimal two-sided ideals.

Now that we have described the Wedderburn decomposition of $H_{4n}(Z)$ and given an explicit construction of the irreducible representations of W , we take up the question of finding formulas for the action of a simple reflection.

For x in \mathfrak{N} , formulas for the action of a simple reflection on the basis of $H_{2d_x}(\mathcal{B}_x)$ given by the irreducible components were first given by Hotta and then refined by Borho, Brylinski, and MacPherson (see [Hot85] and [BBM89, §4.14]). Analogous formulas for the action of a simple reflection on $H_{4n}(Z)$ have been given by Hinich and Joseph [HJ05, §5]. The first two parts of the next theorem may be recovered from the more general (and more complicated) argument in [DR08a, §5].

Theorem 3.9. *Suppose that s is a simple reflection in W and w is in W .*

- (a) $\lambda_s = [\overline{Z_s}] + 1$.
- (b) If $sw < w$, then $[\overline{Z_s}] * [\overline{Z_w}] = -2[\overline{Z_w}]$.
- (c) If $sw > w$, then there is a subset $F_{s,w}$ of $\{x \in W \mid x < w, sx < x\}$ so that $[\overline{Z_s}] * [\overline{Z_w}] = [\overline{Z_{sw}}] + \sum_{x \in F_{s,w}} n_x [\overline{Z_x}]$ with $n_x > 0$.

Using this result, Hinich and Joseph [HJ05, Theorem 5.5] prove a result analogous to Proposition 3.7 for right Steinberg cells. Recall that for w in W we have defined $\mathfrak{V}_r(w) = \overline{Bu_w} \cap \mathfrak{D}$ when w is in $W_{\mathfrak{D}}$. For an orbital variety \mathfrak{V} , define $W_{\overline{\mathfrak{V}}} = \{y \in W \mid \mathfrak{V}_r(y) \subseteq \mathfrak{V}\}$.

Theorem 3.10. *For w in W , the smallest subset, S , of W with the property that $[\overline{Z_w}] * \lambda_y$ is in the span of $\{[\overline{Z_x}] \mid x \in S\}$ for all y in W is $\overline{\mathfrak{V}_r(w)}$. In particular, if \mathfrak{V} is any orbital variety, then the span of $\{[\overline{Z_x}] \mid x \in W_{\overline{\mathfrak{V}}}\}$ is a right ideal in $H_{4n}(Z)$.*

3.6. Sheaf-theoretic decomposition of $H_{4n}(Z)$ and $H_i(\mathcal{B}_x)$. For a variety X , the \mathbb{Q} -vector space $H_i(X)$ has more a sophisticated alternate description in terms of sheaf cohomology (see [CG97, §8.3]). The properties of sheaves and perverse sheaves we use in this section may be found in [KS90, Chapter 2,3], [Dim04] and [Bor84].

Let $D(X)$ denote the full subcategory of the derived category of sheaves of \mathbb{Q} -vector spaces on X consisting of complexes with bounded, constructible cohomology. If $f: X \rightarrow Y$ is a morphism, then there are functors

$$Rf_*: D(X) \rightarrow D(Y), \quad Rf_!: D(X) \rightarrow D(Y), \quad f^*: D(Y) \rightarrow D(X), \quad \text{and} \quad f^!: D(Y) \rightarrow D(X).$$

The pair of functors (f^*, Rf_*) is an adjoint pair, as is $(Rf_!, f^!)$. If f is proper, then $Rf_! = Rf_*$ and if f is smooth, then $f^! = f^*[2 \dim X]$.

We consider the constant sheaf, \mathbb{Q}_X , as a complex in $D(X)$ concentrated in degree zero. The dualizing sheaf, \mathbb{D}_X , of X is defined by $\mathbb{D}_X = a_X^! \mathbb{Q}_{\{\text{pt}\}}$, where $a_X: X \rightarrow \{\text{pt}\}$. If X is a rational homology manifold, in particular, if X is smooth, then $\mathbb{D}_X \cong \mathbb{Q}_X[2 \dim X]$ in $D(X)$. It follows from the definitions and because f^* and $f^!$ are functors that if $f: X \rightarrow Y$, then

$$(3.11) \quad \mathbb{Q}_X \cong f^* \mathbb{Q}_Y \quad \text{and} \quad \mathbb{D}_X \cong f^! \mathbb{D}_Y$$

in $D(X)$.

The cohomology and Borel-Moore homology of X have very convenient descriptions in sheaf-theoretic terms:

$$(3.12) \quad H^i(X) \cong \text{Ext}_{D(X)}^i(\mathbb{Q}_X, \mathbb{Q}_X) \quad \text{and} \quad H_i(X) \cong \text{Ext}_{D(X)}^{-i}(\mathbb{Q}_X, \mathbb{D}_X)$$

where for \mathcal{F} and \mathcal{G} in $D(X)$, $\text{Ext}_{D(X)}^i(\mathcal{F}, \mathcal{G}) = \text{Hom}_{D(X)}(\mathcal{F}, \mathcal{G}[i])$.

Now suppose that $f_i: M_i \rightarrow N$ is a proper morphism for $i = 1, 2, 3$ and that $d_2 = \dim M_2$. In contrast to our assumptions in the convolution setup from §3.1 where the M_i were assumed

to be smooth, in the following computation we assume only that M_2 is a rational homology manifold. Consider the cartesian diagram

$$\begin{array}{ccc} M_1 \times_N M_2 & \xrightarrow{f_{1,2}} & N \\ \delta_1 \downarrow & & \downarrow \delta \\ M_1 \times M_2 & \xrightarrow{f_1 \times f_2} & N \times N \end{array}$$

where $f_{1,2}$ is the induced map. Using the argument in [CG97, §8.6], we have isomorphisms

$$H_i(M_1 \times_N M_2) \cong \mathrm{Ext}_{D(M_1 \times_N M_2)}^{-i}(\mathbb{Q}_{M_1 \times_N M_2}, \mathbb{D}_{M_1 \times_N M_2}) \quad (3.12)$$

$$\cong \mathrm{Ext}_{D(M_1 \times_N M_2)}^{-i}(f_{1,2}^* \mathbb{Q}_N, \delta_1^! \mathbb{D}_{M_1 \times M_2}) \quad (3.11)$$

$$\cong \mathrm{Ext}_{D(N)}^{-i}(\mathbb{Q}_N, R(f_{1,2})_* \delta_1^! \mathbb{D}_{M_1 \times M_2}) \quad (\text{adjunction})$$

$$\cong \mathrm{Ext}_{D(N)}^{-i}(\mathbb{Q}_N, \delta^! R(f_1 \times f_2)_* \mathbb{D}_{M_1 \times M_2}) \quad (\text{base change})$$

$$\cong \mathrm{Ext}_{D(N)}^{-i}(\mathbb{Q}_N, \delta^! (R(f_1)_* \mathbb{D}_{M_1} \boxtimes R(f_2)_* \mathbb{D}_{M_2})) \quad (\text{K\"unneth})$$

$$\cong \mathrm{Ext}_{D(N)}^{-i}(\mathbb{Q}_N, \mathcal{H}om(R(f_1)_* \mathbb{Q}_{M_1}, R(f_2)_* \mathbb{D}_{M_2})) \quad ([\text{Bor84}, 10.25])$$

$$\cong \mathrm{Ext}_{D(N)}^{-i}(\mathbb{Q}_N, \mathcal{H}om(R(f_1)_* \mathbb{Q}_{M_1}, R(f_2)_* \mathbb{Q}_{M_2}[2d_2])) \quad (\mathbb{D}_{M_2} \cong \mathbb{Q}_{M_2}[2d_2])$$

$$\cong \mathrm{Ext}_{D(N)}^{2d_2-i}(\mathbb{Q}_N, \mathcal{H}om(R(f_1)_* \mathbb{Q}_{M_1}, R(f_2)_* \mathbb{Q}_{M_2}))$$

$$\cong \mathrm{Ext}_{D(N)}^{2d_2-i}(R(f_1)_* \mathbb{Q}_{M_1}, R(f_2)_* \mathbb{Q}_{M_2}).$$

Let $\epsilon_{1,2}$ denote the composition of the above isomorphisms, so

$$(3.13) \quad \epsilon_{1,2}: H_i(M_1 \times_N M_2) \xrightarrow{\cong} \mathrm{Ext}_{D(N)}^{2d_2-i}(R(f_1)_* \mathbb{Q}_{M_1}, R(f_2)_* \mathbb{Q}_{M_2}).$$

Chriss and Ginzburg [CG97, §8.6] have shown that the isomorphisms $\epsilon_{1,2}$ intertwine the convolution product on the left with the Yoneda product (composition of morphisms) on the right: given c in $H_i(M_1 \times_N M_2)$ and d in $H_j(M_2 \times_N M_3)$, we have $\epsilon_{1,3}(c * d) = \epsilon_{2,3}(d) \circ \epsilon_{1,2}(c)$.

We may apply the computation in equation (3.13) to $H_*(Z)$. We have seen that $Z \cong \tilde{\mathfrak{N}} \times_{\mathfrak{N}} \tilde{\mathfrak{N}}$ and so

$$H_i(Z) \cong \mathrm{Ext}_{D(\mathfrak{N})}^{4n-i}(R\mu_* \mathbb{Q}_{\tilde{\mathfrak{N}}}, R\mu_* \mathbb{Q}_{\tilde{\mathfrak{N}}}).$$

In particular, taking $i = 4n$, we conclude that are algebra isomorphisms

$$\mathbb{Q}[W] \cong H_{4n}(Z) \cong \mathrm{End}_{D(\mathfrak{N})}(R\mu_* \mathbb{Q}_{\tilde{\mathfrak{N}}})^{\mathrm{op}}.$$

The category $D(\mathfrak{N})$ is a triangulated category. It contains a full, abelian subcategory, denoted by $\mathcal{M}(\mathfrak{N})$, consisting of “perverse sheaves on \mathfrak{N} ” (with respect to the middle perversity). It follows from the Decomposition Theorem of Beilinson, Bernstein, and Deligne [BBD82, §5] that the complex $R\mu_* \mathbb{Q}_{\tilde{\mathfrak{N}}}$ is a semisimple object in $\mathcal{M}(\mathfrak{N})$.

The simple objects in $\mathcal{M}(\mathfrak{N})$ have a geometric description. Suppose X is a smooth, locally closed subvariety of \mathfrak{N} with codimension d , $i: X \rightarrow \mathfrak{N}$ is the inclusion, and L is an irreducible local system on X . Let $\mathrm{IC}(X, L)$ denote the intersection complex of Goresky and MacPherson [GM83, §3]. Then $i_* \mathrm{IC}(\overline{X}, L)[-2d]$ is a simple object in $\mathcal{M}(\mathfrak{N})$ and every simple object arises in this way. In addition to the original sources, [BBD82] and [GM83], we refer the reader to [Sho88, §3] and [CG97, §8.4] for short introductions to the theory of

intersection complexes and perverse sheaves and to [Bor84] and [Dim04] for more thorough expositions.

Returning to $R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}}$, Borho and MacPherson [BM81] have shown that its decomposition into simple perverse sheaves is given by

$$(3.14) \quad R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}} \cong \bigoplus_{x,\phi} j_*^x \mathrm{IC}(\overline{Gx}, L_\phi)[-2d_x]^{n_{x,\phi}}$$

where x runs over the set of orbit representatives \mathfrak{S} in \mathfrak{N} , and for each x , $j^x: \overline{Gx} \rightarrow \mathfrak{N}$ is the inclusion, ϕ is in $\widehat{C(x)}$, L_ϕ is the local system on Gx corresponding to ϕ , and $n_{x,\phi}$ is a non-negative integer.

Define $\mathrm{IC}_{x,\phi} = j_*^x \mathrm{IC}(\overline{Gx}, L_\phi)$. Then $\mathrm{IC}_{x,\phi}[-2d_x]$ is a simple object in $\mathcal{M}(\mathfrak{N})$. It follows from the computation of the groups $C(x)$ that $\mathrm{End}_{D(\mathfrak{N})}(\mathrm{IC}_{x,\phi}) \cong \mathbb{Q}$. Therefore,

$$(3.15) \quad \begin{aligned} H_{4n}(Z) &\cong \mathrm{End}_{D(\mathfrak{N})}(R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}})^{\mathrm{op}} \\ &\cong \mathrm{End}_{D(\mathfrak{N})}(\bigoplus_{x,\phi} \mathrm{IC}_{x,\phi}[-2d_x]^{n_{x,\phi}})^{\mathrm{op}} \\ &\cong \bigoplus_{x,\phi} \mathrm{End}_{D(\mathfrak{N})}(\mathrm{IC}_{x,\phi}^{n_{x,\phi}})^{\mathrm{op}} \\ &\cong \bigoplus_{x,\phi} M_{n_{x,\phi}}(\mathrm{End}_{D(\mathfrak{N})}(\mathrm{IC}_{x,\phi}))^{\mathrm{op}} \\ &\cong \bigoplus_{x,\phi} M_{n_{x,\phi}}(\mathbb{Q})^{\mathrm{op}}. \end{aligned}$$

This is a decomposition of $H_{4n}(Z)$ as a direct sum of matrix rings and hence is the Wedderburn decomposition of $H_{4n}(Z)$.

Suppose now that \mathfrak{D} is a G -orbit in \mathfrak{N} and x is in \mathfrak{D} . It is straightforward to check that

$$H_{\mathfrak{D}} \cong \bigoplus_{\phi \in \widehat{C(x)}} \mathrm{End}_{D(\mathfrak{N})}((\mathrm{IC}_{x,\phi})^{n_{x,\phi}}) \cong \bigoplus_{\phi \in \widehat{C(x)}} M_{n_{x,\phi}}(\mathbb{Q}).$$

As in Proposition 3.7, this is the decomposition of $H_{\mathfrak{D}}$ into minimal two-sided ideals.

For a second application of (3.13), let $i_x: \{x\} \rightarrow \mathfrak{N}$ denote the inclusion. Then $\mathcal{B}_x \cong \tilde{\mathfrak{N}} \times_{\mathfrak{N}} \{x\}$ and so

$$\begin{aligned} H_i(\mathcal{B}_x) &\cong \mathrm{Ext}_{D(\mathfrak{N})}^{-i}(R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}}, R(i_x)_*\mathbb{Q}_{\{x\}}) \\ &\cong \bigoplus_{y,\psi} \mathrm{Ext}_{D(\mathfrak{N})}^{-i}(\mathrm{IC}_{y,\psi}[-2d_y]^{n_{y,\psi}}, R(i_x)_*\mathbb{Q}_{\{x\}}) \\ &\cong \bigoplus_{y,\psi} \mathrm{Ext}_{D(\mathfrak{N})}^{2d_y-i}(\mathrm{IC}_{y,\psi}^{n_{y,\psi}}, R(i_x)_*\mathbb{Q}_{\{x\}}) \\ &\cong \bigoplus_{y,\psi} \left(V_{y,\psi} \otimes \mathrm{Ext}_{D(\mathfrak{N})}^{2d_y-i}(\mathrm{IC}_{y,\psi}, R(i_x)_*\mathbb{Q}_{\{x\}}) \right) \end{aligned}$$

where $V_{y,\psi}$ is an $n_{y,\psi}$ -dimensional vector space. Because $\mathbb{Q}[W] \cong H_{4n}(Z) \cong \mathrm{End}_{D(\mathfrak{N})}(R\mu_*\mathbb{Q}_{\tilde{\mathfrak{N}}})$ acts by permuting the simple summands, it follows from (3.15) that each $V_{y,\psi}$ affords an irreducible representation of W and that $\mathrm{Ext}_{D(\mathfrak{N})}^{2d_y-i}(\mathrm{IC}_{y,\psi}, R(i_x)_*\mathbb{Q}_{\{x\}})$ records the multiplicity of $V_{y,\psi}$ in $H_i(\mathcal{B}_x)$. Using that i_x^* is left adjoint to $R(i_x)_*$, denoting the stalk of $\mathrm{IC}_{y,\psi}$ at x by

$(\mathrm{IC}_{y,\psi})_x$, and setting $m_{y,\psi}^{x,i} = \dim \mathrm{Ext}_{D(\mathfrak{N})}^{2d_y-i}(\mathrm{IC}_{y,\psi}, R(i_x)_* \mathbb{Q}_{\{x\}})$, we obtain the decomposition of $H_i(\mathcal{B}_x)$ into irreducible representations of W :

$$H_i(\mathcal{B}_x) \cong \bigoplus_{y,\psi} \left(V_{y,\psi} \otimes \mathrm{Ext}_{D(\{x\})}^{2d_y-i}((\mathrm{IC}_{y,\psi})_x, \mathbb{Q}_{\{x\}}) \right) \cong \bigoplus_{y,\psi} V_{y,\psi}^{m_{y,\psi}^{x,i}}.$$

In the next subsection we apply (3.13) to compute the Borel-Moore homology of some generalized Steinberg varieties.

3.7. Borel-Moore homology of generalized Steinberg varieties. Recall from §2.4 the generalized Steinberg variety

$$X^{\mathcal{P},\mathcal{Q}} = \{ (x, P', Q') \in \mathfrak{N} \times \mathcal{P} \times \mathcal{Q} \mid x \in \mathfrak{p}' \cap \mathfrak{q}' \} \cong \tilde{\mathfrak{N}}^{\mathcal{P}} \times_{\mathfrak{N}} \tilde{\mathfrak{N}}^{\mathcal{Q}}$$

where $\tilde{\mathfrak{N}}^{\mathcal{P}} = \{ (x, P') \in \mathfrak{N} \times \mathcal{P} \mid x \in \mathfrak{p}' \}$, $\xi^{\mathcal{P}}: \tilde{\mathfrak{N}}^{\mathcal{P}} \rightarrow \mathfrak{N}$ is projection on the first factor, and $\tilde{\mathfrak{N}}^{\mathcal{Q}}$ and $\xi^{\mathcal{Q}}$ are defined similarly. Recall also that $\eta: Z \rightarrow X^{\mathcal{P},\mathcal{Q}}$ is a proper, G -equivariant surjection. The main result of [DR08a, Theorem 4.4], which is proved using the constructions in the last subsection, is the following theorem describing the Borel-Moore homology of $X^{\mathcal{P},\mathcal{Q}}$.

Theorem 3.16. *Consider $H_{4n}(Z)$ as a $W \times W$ -module using the isomorphism $H_{4n}(Z) \cong \mathbb{Q}[W]$. Then there is an isomorphism $\alpha: H_*(X^{\mathcal{P},\mathcal{Q}}) \xrightarrow{\cong} H_*(Z)^{W_P \times W_Q}$ so that the composition $\alpha \circ \eta_*: H_*(Z) \rightarrow H_*(Z)^{W_P \times W_Q}$ is the averaging map.*

As a special case of the theorem, if we let e_P (resp. e_Q) denote the primitive idempotent in $\mathbb{Q}[W_P]$ (resp. $\mathbb{Q}[W_Q]$) corresponding to the trivial representation, then

$$(3.17) \quad H_{4n}(X^{\mathcal{P},\mathcal{Q}}) \cong e_P \mathbb{Q}[W] e_Q.$$

Next recall the generalized Steinberg variety $X_{0,0}^{\mathcal{P},\mathcal{Q}} \cong T^*\mathcal{P} \times_{\mathfrak{N}} T^*\mathcal{Q}$. Set $m = \dim P/B + \dim Q/B$. Let ϵ_P (resp. ϵ_Q) denote the primitive idempotent in $\mathbb{Q}[W_P]$ (resp. $\mathbb{Q}[W_Q]$) corresponding to the sign representation. Then $\dim X_{0,0}^{\mathcal{P},\mathcal{Q}} = 4n - 2m$ and it is shown in [DR08a, §5] that

$$(3.18) \quad H_{4n-2m}(X_{0,0}^{\mathcal{P},\mathcal{Q}}) \cong \epsilon_P \mathbb{Q}[W] \epsilon_Q.$$

Now suppose that c is a Levi class function on \mathcal{P} . Let L be a Levi subgroup of P and choose x in $c(P) \cap \mathfrak{l}$. Then we may consider the Springer representation of W_P on $H_{2d_x^L}(\mathcal{B}_x^L)^{C_L(x)}$ where $C_L(x)$ is the component group of $Z_L(x)$, \mathcal{B}_x^L is the variety of Borel subalgebras of \mathfrak{l} that contain x , and $d_x^L = \dim \mathcal{B}_x^L$. This is an irreducible representation of W_P . Let f_P denote a primitive idempotent in $\mathbb{Q}[W_P]$ so that $\mathbb{Q}[W_P]f_P \cong H_{2d_x^L}(\mathcal{B}_x^L)^{C_L(x)}$. Set $\delta_{c,d}^{\mathcal{P},\mathcal{Q}} = \frac{1}{2}(\dim c(P) + \dim \mathfrak{u}_P + \dim d(Q) + \dim \mathfrak{u}_Q)$. Then it is shown in [DR04, Corollary 2.6] that $\dim X_{c,d}^{\mathcal{P},\mathcal{Q}} \leq \delta_{c,d}^{\mathcal{P},\mathcal{Q}}$. Generalizing the computations (3.17) and (3.18), we conjecture that the following statement is true.

Conjecture 3.19. *With the notation above, $H_{\delta_{c,d}^{\mathcal{P},\mathcal{Q}}}(X_{c,d}^{\mathcal{P},\mathcal{Q}}) \cong f_P \mathbb{Q}[W] f_Q$.*

The Borel-Moore homology of $X^{\mathcal{P},\mathcal{Q}}$ may also be computed using the sheaf theoretic methods in the last subsection. We have $X^{\mathcal{P},\mathcal{Q}} \cong \tilde{\mathfrak{N}}^{\mathcal{P}} \times_{\mathfrak{N}} \tilde{\mathfrak{N}}^{\mathcal{Q}}$ and Borho and MacPherson [BM83, 2.11] have shown that $\tilde{\mathfrak{N}}^{\mathcal{P}}$ and $\tilde{\mathfrak{N}}^{\mathcal{Q}}$ are rational homology manifolds. Therefore, as in (3.13):

$$H_i(X^{\mathcal{P},\mathcal{Q}}) \cong \mathrm{Ext}_{D(\mathfrak{N})}^{4n-i}(R\xi_*^{\mathcal{P}} \mathbb{Q}_{\tilde{\mathfrak{N}}^{\mathcal{P}}}, R\xi_*^{\mathcal{Q}} \mathbb{Q}_{\tilde{\mathfrak{N}}^{\mathcal{Q}}}).$$

Borho and MacPherson [BM83, 2.11] have also shown that $R\xi_*^{\mathcal{P}}\mathbb{Q}_{\tilde{\mathfrak{N}}^{\mathcal{P}}}$ is a semisimple object in $\mathcal{M}(\mathfrak{N})$ and described its decomposition into simple perverse sheaves:

$$R\xi_*^{\mathcal{P}}\mathbb{Q}_{\tilde{\mathfrak{N}}^{\mathcal{P}}} \cong \bigoplus_{(x,\phi)} \mathrm{IC}_{x,\phi}[-2d_x]^{n_{x,\phi}^{\mathcal{P}}},$$

where the sum is over pairs (x, ϕ) as in equation (3.14), and $n_{x,\phi}^{\mathcal{P}}$ is the multiplicity of the irreducible representation $H_{2d_x}(\mathcal{B}_x)_{\phi}$ of W in the induced representation $\mathrm{Ind}_{W_P}^W(1_{W_P})$. Thus,

$$H_i(X^{\mathcal{P},\mathcal{Q}}) \cong \bigoplus_{x,\phi} \bigoplus_{y,\psi} \mathrm{Ext}_{D(\mathfrak{N})}^{4n-i} \left(\mathrm{IC}_{x,\phi}[-2d_x]^{n_{x,\phi}^{\mathcal{P}}}, \mathrm{IC}_{y,\psi}[-2d_y]^{n_{y,\psi}^{\mathcal{Q}}} \right)$$

and so

$$(3.20) \quad H_{4n}(X^{\mathcal{P},\mathcal{Q}}) \cong \bigoplus_{x,\phi} \bigoplus_{y,\psi} \mathrm{Hom}_{D(\mathfrak{N})} \left(\mathrm{IC}_{x,\phi}[-2d_x]^{n_{x,\phi}^{\mathcal{P}}}, \mathrm{IC}_{y,\psi}[-2d_y]^{n_{y,\psi}^{\mathcal{Q}}} \right) \cong \bigoplus_{x,\phi} M_{n_{x,\phi}^{\mathcal{Q}}, n_{x,\phi}^{\mathcal{P}}}(\mathbb{Q}).$$

Using the fact that $n_{x,\phi}^{\mathcal{P}}$ is the multiplicity of the irreducible representation $H_{2d_x}(\mathcal{B}_x)_{\phi}$ of W in the induced representation $\mathrm{Ind}_{W_P}^W(1_{W_P})$, we see that (3.20) is consistent with (3.17).

4. EQUIVARIANT K -THEORY

Certainly the most important result to date involving the Steinberg variety is its application by Kazhdan and Lusztig to the Langlands program [KL87]. They show that the equivariant K -theory of Z is isomorphic to the two-sided regular representation of the extended, affine Hecke algebra \mathcal{H} . They then use this representation of \mathcal{H} to classify simple \mathcal{H} -modules and hence to classify representations of ${}^L G(\mathbb{Q}_p)$ containing a vector fixed by an Iwahori subgroup, where ${}^L G(\mathbb{Q}_p)$ is the group of \mathbb{Q}_p -points of the Langlands dual of G . As with homology, Chriss and Ginzburg have applied the convolution product formalism to the equivariant K -theory of Z and recast Kazhdan and Lusztig's results as an algebra isomorphism.

Recall we are assuming that G is simply connected. In this section we describe the isomorphism $\mathcal{H} \cong K^{\overline{G}}(Z)$, where $\overline{G} = G \times \mathbb{C}^*$, and we give some applications to the study of nilpotent orbits. We emphasize in particular the relationship between nilpotent orbits, Kazhdan-Lusztig theory for the extended, affine Weyl group, and (generalized) Steinberg varieties.

4.1. The generic, extended, affine Hecke algebra. We begin by describing the Bernstein-Zelevinski presentation of the extended, affine Hecke algebra following the construction in [Lus89a].

Let v be an indeterminate and set $A = \mathbb{Z}[v, v^{-1}]$. The ring A is the base ring of scalars for most of the constructions in this section.

Let $X(T)$ denote the character group of T . Since G is simply connected, $X(T)$ is the weight lattice of G . Define X^+ to be the set of dominant weights with respect to the base of the root system of (G, T) determined by B . The *extended, affine Weyl group* is $W_e = X(T) \rtimes W$.

There is a “length function” ℓ on W_e that extends the usual length function on W . The *braid group of W_e* is the group \mathcal{Br} , with generators $\{T_x \mid x \in W_e\}$ and relations $T_x T_{x'} = T_{xx'}$ if $\ell(x) + \ell(x') = \ell(xx')$. The *generic, extended, affine Hecke algebra*, \mathcal{H} , is the quotient of

the group algebra $A[\mathcal{B}r]$ by the two-sided ideal generated by the elements $(T_s + 1)(T_s - v^2)$, where s runs through the simple reflections in W .

Let ${}^L G$ denote the Langlands dual of G , so ${}^L G$ is an adjoint group. Let ${}^L G_p$ denote the algebraic group over \mathbb{Q}_p with the same type as ${}^L G$. Suppose that I is an Iwahori subgroup of ${}^L G_p$ and let $\mathbb{C}[I \backslash {}^L G_p / I]$ denote the space of all compactly supported functions ${}^L G_p \rightarrow \mathbb{C}$ that are constant on (I, I) -double cosets. Consider \mathbb{C} as an A -module via the specialization $A \rightarrow \mathbb{C}$ with $v \mapsto \sqrt{p}$. The following theorem, due to Iwahori and Matsumoto [IM65, §3], relates \mathcal{H} to representations of ${}^L G_p$ containing an I -fixed vector.

Theorem 4.1. *The (I, I) -double cosets of ${}^L G_p$ are parametrized by W_e . Moreover, if I_x is the double coset indexed by x in W_e , then the map which sends T_x to the characteristic function of I_x extends to an algebra isomorphism*

$$\mathbb{C} \otimes_A \mathcal{H} \cong \mathbb{C}[I \backslash {}^L G_p / I].$$

The algebra \mathcal{H} has a factorization (as a tensor product) analogous to the factorization $W_e = X(T) \rtimes W$. Given λ in $X(T)$ one can write $\lambda = \lambda_1 - \lambda_2$ where λ_1 and λ_2 are in X^+ . Define E^λ in \mathcal{H} to be the image of $v^{\ell(\lambda_1 - \lambda_2)} T_\lambda$. For x in W_e , denote the image of T_x in \mathcal{H} again by T_x . Let \mathcal{H}_W denote the Iwahori-Hecke algebra of W (an A -algebra) with standard basis $\{t_w \mid w \in W\}$. Lusztig [Lus89a, §2] has proved the following theorem.

Theorem 4.2. *With the notation above we have:*

- (a) E^λ does not depend on the choice of λ_1 and λ_2 .
- (b) The mapping $A[X(T)] \otimes_A \mathcal{H}_W \rightarrow \mathcal{H}$ defined by $\lambda \otimes t_w \mapsto E^\lambda T_w$ is an isomorphism of A -modules.
- (c) For λ and λ' in X we have $E^\lambda E^{\lambda'} = E^{\lambda + \lambda'}$ and so the subspace of \mathcal{H} spanned by $\{E^\lambda \mid \lambda \in X\}$ is a subalgebra isomorphic to $A[X(T)]$.
- (d) The center of \mathcal{H} is isomorphic to $A[X(T)]^W$ via the isomorphism in (c).
- (e) The subspace of \mathcal{H} spanned by $\{T_w \mid w \in W\}$ is a subalgebra isomorphic to \mathcal{H}_W .

Using parts (b) and (d) of the theorem, we identify $A[X(T)]$ with the subalgebra of \mathcal{H} spanned by $\{E^\lambda \mid \lambda \in X\}$, and $A[X(T)]^W$ with the center of \mathcal{H} .

4.2. Equivariant K -theory and convolution. Two introductory references for the notions from equivariant K -theory we use are [BBM89, Chapter 2] and [CG97, Chapter 5].

For a variety X , let $\mathcal{Coh}(X)$ denote the category of coherent \mathcal{O}_X -modules. Suppose that H is a linear algebraic group acting on X . Let $a: H \times X \rightarrow X$ be the action morphism and $p: H \times X \rightarrow X$ be the projection. An H -equivariant coherent \mathcal{O}_X -module is a pair (\mathcal{M}, i) , where \mathcal{M} is a coherent \mathcal{O}_X -module and $i: a^* \mathcal{M} \xrightarrow{\sim} p^* \mathcal{M}$ is an isomorphism satisfying several conditions (see [CG97, §5.1] for the precise definition). With the obvious notion of morphism, H -equivariant \mathcal{O}_X -modules form an abelian category denoted by $\mathcal{Coh}^H(X)$. The Grothendieck group of $\mathcal{Coh}^H(X)$ is denoted by $K^H(X)$ and is called the H -equivariant K -group of X .

If $X = \{\text{pt}\}$ is a point, then $K^H(\text{pt}) \cong R(H)$ is the representation ring of H . For any X , $K^H(X)$ is naturally an $R(H)$ -module. If H is the trivial group, then $\mathcal{Coh}^H(X) = \mathcal{Coh}(X)$ and $K^H(X) = K(X)$ is the Grothendieck group of the category of coherent \mathcal{O}_X -modules.

As with Borel-Moore homology, equivariant K -theory is a bivariant theory in the sense of Fulton and MacPherson [FM81]: Suppose that X and Y are H -varieties and that $f: X \rightarrow Y$

is an H -equivariant morphism. If f is proper, there is a direct image map in equivariant K -theory, $f_*: K^H(X) \rightarrow K^H(Y)$, and if f is smooth there is a pullback map $f^*: K^H(Y) \rightarrow K^H(X)$ in equivariant K -theory. Moreover, if X is smooth and A and B are closed, H -stable subvarieties of X , there is an intersection pairing $\cap: K^H(A) \times K^H(B) \rightarrow K^H(A \cap B)$ (called a Tor-product in [Lus98, §6.4]). This pairing depends on (X, A, B) . Thus, we may apply the convolution product construction from §3.1 in the equivariant K -theory setting.

In more detail, suppose that for $i = 1, 2, 3$, M_i is a smooth variety with an algebraic action of H and $f_i: M_i \rightarrow N$ is a proper, H -equivariant morphism. Suppose that for $1 \leq i < j \leq 3$, $Z_{i,j}$ is a closed, H -stable subvariety of $M_i \times M_j$ and that $p_{1,3}: p_{1,2}^{-1}(Z_{1,2}) \cap p_{2,3}^{-1}(Z_{2,3}) \rightarrow Z_{1,3}$ is a proper morphism. Then as in §3.1, the formula $c * d = (p_{1,3})_*(p_{1,2}^*(c) \cap p_{2,3}^*(d))$, where \cap is the intersection pairing determined by the subsets $Z_{1,2} \times M_3$ and $M_1 \times Z_{2,3}$ of $M_1 \times M_2 \times M_3$, defines an associative convolution product, $K^H(Z_{1,2}) \otimes K^H(Z_{2,3}) \xrightarrow{*} K^H(Z_{1,3})$.

In particular, the convolution product defines a ring structure on $K^G(Z)$. It is shown in [CG97, Theorem 7.2.2] that with this ring structure, $K^G(Z)$ is isomorphic to the group ring $\mathbb{Z}[W_e]$. In the next subsection we describe a more general result with $\mathbb{Z}[W_e]$ replaced by \mathcal{H} and G replaced by $G \times \mathbb{C}^*$, where \mathbb{C}^* denote the multiplicative group of non-zero complex numbers.

The variable, v , in the definition of \mathcal{H} is given a geometric meaning using the isomorphism $X(\mathbb{C}^*) \cong \mathbb{Z}$. Let $1_{\mathbb{C}^*}$ denote the trivial representation of \mathbb{C}^* . Then the rule $v \mapsto 1_{\mathbb{C}^*}$ extends to a ring isomorphism $\mathbb{Z}[v, v^{-1}] \cong R(\mathbb{C}^*)$. For the rest of this paper we will use this isomorphism to identify $A = \mathbb{Z}[v, v^{-1}]$ and $R(\mathbb{C}^*)$.

4.3. The Kazhdan-Lusztig isomorphism. To streamline the notation, set $\overline{G} = G \times \mathbb{C}^*$. Then $R(\overline{G}) \cong R(G) \otimes_{\mathbb{Z}} R(\mathbb{C}^*) \cong R(G) \otimes_{\mathbb{Z}} A = R(G)[v, v^{-1}]$.

Similarly, for a closed subgroup, H , of G , we denote the subgroup $H \times \mathbb{C}^*$ of \overline{G} by \overline{H} . In particular, $\overline{T} = T \times \mathbb{C}^*$ and $\overline{B} = B \times \mathbb{C}^*$. In the remainder of this paper we will never need to consider the closure of a subgroup of G and so this notation should not lead to any confusion.

Define a \mathbb{C}^* -action on \mathfrak{g} by $(\xi, x) \mapsto \xi^2 x$. We consider \mathcal{B} as a \mathbb{C}^* -set with the trivial action. Then the action of G on $\tilde{\mathfrak{N}}$ and Z extends to an action of \overline{G} on $\tilde{\mathfrak{N}}$ and Z , and μ_z and μ are \overline{G} -equivariant.

Recall from §4.1 that we are viewing the group algebra $A[X(T)]$ as a subspace of \mathcal{H} , and that the center of \mathcal{H} is $Z(\mathcal{H}) = A[X(T)]^W$. Using the identification $A = R(\mathbb{C}^*)$, we may begin to interpret subspaces of \mathcal{H} in K -theoretic terms:

$$K^{\overline{G}}(\{\text{pt}\}) \cong R(\overline{G}) \cong R(G) \otimes R(\mathbb{C}^*) \cong R(G)[v, v^{-1}] \cong A[X(T)]^W = Z(\mathcal{H}).$$

Recall that the “diagonal” subvariety, Z_1 , of the Steinberg variety is defined by $Z_1 = \{(x, B', B') \in \mathfrak{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathfrak{b}'\}$. For suitable choices of $f_i: M_i \rightarrow N$ and $Z_{i,j}$, and using the embedding $A \subseteq R(\overline{G})$, the convolution product induces various A -linear maps:

- (1) $K^{\overline{G}}(Z) \times K^{\overline{G}}(Z) \xrightarrow{*} K^{\overline{G}}(Z)$; with this multiplication, $K^{\overline{G}}(Z)$ is an A -algebra.
- (2) $K^{\overline{G}}(Z_1) \times K^{\overline{G}}(Z_1) \xrightarrow{*} K^{\overline{G}}(Z_1)$; with this multiplication, $K^{\overline{G}}(Z_1)$ is a commutative A -algebra.
- (3) $K^{\overline{G}}(Z) \times K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B}) \xrightarrow{*} K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B})$; this defines a left $K^{\overline{G}}(Z)$ -module structure on $K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B})$.

The group $K^{\overline{G}}(Z_1)$ has a well-known description. First, the rule $(x, B') \mapsto (x, B', B')$ defines a \overline{G} -equivariant isomorphism between $\tilde{\mathfrak{N}}$ and Z_1 and hence an isomorphism $K^{\overline{G}}(Z_1) \cong K^{\overline{G}}(\tilde{\mathfrak{N}})$. Second, the projection $\tilde{\mathfrak{N}} \rightarrow \mathcal{B}$ is a vector bundle and so, using the Thom isomorphism in equivariant K -theory [CG97, §5.4], we have $K^{\overline{G}}(\tilde{\mathfrak{N}}) \cong K^{\overline{G}}(\mathcal{B})$. Third, \mathcal{B} is isomorphic to $G \times^B \{\text{pt}\}$ by a \overline{G} -equivariant isomorphism and so $K^{\overline{G}}(\mathcal{B}) \cong K^{\overline{B}}(\{\text{pt}\}) \cong R(\overline{B})$ by a version of Frobenius reciprocity in equivariant K -theory [CG97, §5.2.16]. Finally, since U is the unipotent radical of B , we have

$$R(\overline{B}) \cong R(\overline{B}/\overline{U}) \cong R(\overline{T}) \cong R(T)[v, v^{-1}] \cong A[X(T)].$$

Composing these isomorphisms, we get an isomorphism $K^{\overline{G}}(Z_1) \xrightarrow{\cong} A[X(T)]$, which is in fact an isomorphism of A -algebras.

The inverse isomorphism $A[X(T)] \xrightarrow{\cong} K^{\overline{G}}(Z_1)$ may be computed explicitly. Suppose that λ is in $X(T)$. Then λ lifts to a representation of \overline{B} . Denote the representation space by \mathbb{C}_λ . Then the projection morphism $\overline{G} \times^{\overline{B}} \mathbb{C}_\lambda \rightarrow \mathcal{B}$ is a \overline{G} -equivariant line bundle on \mathcal{B} . The sheaf of sections of this line bundle is a \overline{G} -equivariant, coherent sheaf of $\mathcal{O}_{\mathcal{B}}$ -modules that we will denote by L_λ . Pulling L_λ back first through the vector bundle projection $\tilde{\mathfrak{N}} \rightarrow \mathcal{B}$ and then through the isomorphism $Z_1 \cong \tilde{\mathfrak{N}}$, we get a \overline{G} -equivariant, coherent sheaf of \mathcal{O}_{Z_1} -modules we denote by \mathcal{L}_λ .

Let $i_1: Z_1 \rightarrow Z$ be the inclusion. Define $e^\lambda = (i_1)_*([\mathcal{L}_\lambda])$ in $K^{\overline{G}}(Z)$. Then $\lambda \mapsto e^\lambda$ defines an A -linear map from $A[X(T)]$ to $K^{\overline{G}}(Z)$.

A concentration theorem due to Thomason and the Cellular Fibration Lemma of Chriss and Ginzburg can be used to prove the following proposition (see [CG97, 6.2.7] and [Lus98, 7.15]).

Proposition 4.3. *The closed embeddings $i_1: Z_1 \rightarrow Z$ and $j: Z \rightarrow \tilde{\mathfrak{N}} \times \mathcal{B}$ induce injective maps in equivariant K -theory,*

$$K^{\overline{G}}(Z_1) \xrightarrow{(i_1)_*} K^{\overline{G}}(Z) \xrightarrow{j_*} K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B}).$$

The map $(i_1)_$ is an A -algebra monomorphism and the map j_* is a $K^{\overline{G}}(Z)$ -module monomorphism. In particular, $K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B})$ is a faithful $K^{\overline{G}}(Z)$ -module.*

From the proposition and the isomorphism $K^{\overline{G}}(\{\text{pt}\}) \cong Z(\mathcal{H})$, we see that there is a commutative diagram of A -algebras and A -algebra homomorphisms:

$$\begin{array}{ccccc} Z(\mathcal{H}) & \hookrightarrow & A[X(T)] & \hookrightarrow & \mathcal{H} \\ \cong \downarrow & & \cong \downarrow & & \\ K^{\overline{G}}(\{\text{pt}\}) & \hookrightarrow & K^{\overline{G}}(Z_1) & \hookrightarrow & K^{\overline{G}}(Z). \end{array}$$

We will complete this diagram with an isomorphism of A -algebras $K^{\overline{G}}(Z) \cong \mathcal{H}$ following the argument in [Lus98, §7].

Fix a simple reflection, s , in W . Then there is a simple root, α , in $X(T)$ and a corresponding cocharacter, $\check{\alpha}: \mathbb{C}^* \rightarrow T$, so that if $\langle \cdot, \cdot \rangle$ is the pairing between characters and cocharacters of T , then $\langle \alpha, \check{\alpha} \rangle = 2$ and $s(\lambda) = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha$ for λ in $X(T)$. Choose a weight λ' in $X(T)$ with $\langle \lambda', \check{\alpha} \rangle = -1$ and set $\lambda'' = -\lambda' - \alpha$. Then $L_{\lambda'} \boxtimes L_{\lambda''}$ is in $\mathcal{Coh}^{\overline{G}}(\mathcal{B} \times \mathcal{B})$. Lusztig

[Lus98, 7.19] has shown that the restriction of $L_{\lambda'} \boxtimes L_{\lambda''}$ to the closed subvariety $\overline{G(B, sBs)}$ does not depend on the choice of λ' . Denote the restriction of $L_{\lambda'} \boxtimes L_{\lambda''}$ to $\overline{G(B, sBs)}$ by \mathcal{L}_s .

It is easy to check that $Z_1 \cap \overline{Z_s} = \{(x, gBg^{-1}, gBg^{-1}) \in Z_1 \mid g^{-1}x \in \mathfrak{u}_s\}$. It follows that $\overline{Z_s}$ is smooth and that $\pi: \overline{Z_s} \rightarrow \overline{G(B, sBs)}$ is a vector bundle projection with fibre \mathfrak{u}_s . Thus, there is a pullback map in equivariant K -theory, $\pi^*: K^{\overline{G}}(\overline{G(B, sBs)}) \rightarrow K^{\overline{G}}(\overline{Z_s})$, and so we may consider $\pi^*([\mathcal{L}_s])$ in $K^{\overline{G}}(\overline{Z_s})$. Let $i_s: \overline{Z_s} \rightarrow Z$ denote the inclusion. Then i_s is a closed embedding and so there is a direct image map $(i_s)_*: K^{\overline{G}}(\overline{Z_s}) \rightarrow K^{\overline{G}}(Z)$. Define $l_s = (i_s)_* \pi^*([\mathcal{L}_s])$. Then l_s is in $K^{\overline{G}}(Z)$.

Lusztig [Lus98, 7.24] has proved the following lemma.

Lemma 4.4. *There is a unique left \mathcal{H} -module structure on $K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B})$ with the property that for every k in $K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B})$, λ in $X(T)$, and simple reflection s in W we have*

- (a) $-(T_s + 1) \cdot k = l_s * k$ and
- (b) $E^\lambda \cdot k = e^\lambda * k$.

Now the \mathcal{H} -module and $K^{\overline{G}}(Z)$ -module structures on $K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B})$ determine A -linear ring homomorphisms $\phi_1: \mathcal{H} \rightarrow \text{End}_A(K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B}))$ and $\phi_2: K^{\overline{G}}(Z) \rightarrow \text{End}_A(K^{\overline{G}}(\tilde{\mathfrak{N}} \times \mathcal{B}))$ respectively. It follows from Lemma 4.4 that the image of ϕ_1 is contained in the image of ϕ_2 and it follows from Proposition 4.3 that ϕ_2 is an injection. Therefore, $\phi_2^{-1} \circ \phi_1$ determines an A -algebra homomorphism from \mathcal{H} to $K^{\overline{G}}(Z)$ that we denote by ϕ .

The following theorem is proved in [Lus98, §8] using a construction that goes back to [KL87].

Theorem 4.5. *The A -algebra homomorphism $\phi: \mathcal{H} \rightarrow K^{\overline{G}}(Z)$ is an isomorphism and*

$$\begin{array}{ccccc} Z(\mathcal{H}) & \hookrightarrow & A[X(T)] & \hookrightarrow & \mathcal{H} \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow \phi \\ K^{\overline{G}}(\{\text{pt}\}) & \hookrightarrow & K^{\overline{G}}(Z_1) & \hookrightarrow & K^{\overline{G}}(Z) \end{array}$$

is a commutative diagram of A -algebras and A -algebra homomorphisms.

In [CG97, §7.6] Chriss and Ginzburg construct an isomorphism $\mathcal{H} \cong K^{\overline{G}}(Z)$ that satisfies the conclusions of Theorem 4.5 using a variant of the ideas above.

Set $e = \sum_{w \in W} T_w$ in \mathcal{H} . It is easy to check that there is an A -module isomorphism $K^{\overline{G}}(\tilde{\mathfrak{N}}) \cong \mathcal{H}e$ and hence an A -algebra isomorphism $\text{End}_A(K^{\overline{G}}(\tilde{\mathfrak{N}})) \cong \text{End}_A(\mathcal{H}e)$. The convolution product construction can be used to define the structure of a left $K^{\overline{G}}(Z)$ -module on $K^{\overline{G}}(\tilde{\mathfrak{N}})$ [CG97, §5.4] and hence an A -algebra homomorphism $K^{\overline{G}}(Z) \rightarrow \text{End}_A(K^{\overline{G}}(\tilde{\mathfrak{N}}))$. Similarly, the left \mathcal{H} -module structure on $\mathcal{H}e$ defines an A -algebra homomorphism $\mathcal{H} \rightarrow \text{End}_A(\mathcal{H}e)$. Chriss and Ginzburg show that the diagram

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \text{End}_A(\mathcal{H}e) \\ & & \downarrow \cong \\ K^{\overline{G}}(Z) & \longrightarrow & \text{End}_A(K^{\overline{G}}(\tilde{\mathfrak{N}})) \end{array}$$

can be completed to a commutative square of A -algebras and that the resulting A -algebra homomorphism $\mathcal{H} \rightarrow K^{\overline{G}}(Z)$ is an isomorphism. We will see in §4.5 how this construction leads to a conjectural description of the equivariant K -theory of the generalized Steinberg varieties $X^{\mathcal{P}, \mathcal{Q}}$.

4.4. Irreducible representations of \mathcal{H} , two-sided cells, and nilpotent orbits. The isomorphism in Theorem 4.5 has been used by Kazhdan and Lusztig [KL87, §7] to give a geometric construction and parametrization of irreducible \mathcal{H} -modules. Using this construction, Lusztig [Lus89b, §4] has found a bijection between the set of two-sided Kazhdan-Lusztig cells in W_e and the set of G -orbits in \mathfrak{N} . In order to describe this bijection, as well as a conjectural description of two-sided ideals in $K^{\overline{G}}(Z)$ analogous to the decomposition of $H_{4n}(Z)$ given in Proposition 3.7, we need to review the Kazhdan-Lusztig theory of two-sided cells and Lusztig's based ring J .

The rules $v \mapsto v^{-1}$ and $T_x \mapsto T_{x^{-1}}^{-1}$, for x in W_e , define a ring involution of \mathcal{H} denoted by $h \mapsto \overline{h}$. The argument given by Kazhdan and Lusztig in the proof of [KL79, Theorem 1.1] applies to \mathcal{H} and shows that there is a unique basis, $\{c'_y \mid y \in W_e\}$, of \mathcal{H} with the following properties:

- (1) $\overline{c'_y} = c'_y$ for all y in W_e ; and
- (2) if we write $c'_y = v^{-\ell(y)} \sum_{x \in W_e} P_{x,y} c'_x$, then $P_{y,y} = 1$, $P_{x,y} = 0$ unless $x \leq y$, and $P_{x,y}$ is a polynomial in v^2 with degree (in v) at most $\ell(y) - \ell(x) - 1$ when $x < y$.

The polynomials $P_{x,y}$ are called *Kazhdan-Lusztig polynomials*.

For x and y in W_e , define $x \leq_{LR} y$ if there exists h_1 and h_2 in \mathcal{H} so that when $h_1 c'_y h_2$ is expressed as a linear combination of c'_z , the coefficient of c'_x is non-zero. It follows from the results in [KL79, §1] that \leq_{LR} is a preorder on W_e . The equivalence classes determined by this preorder are *two-sided Kazhdan-Lusztig cells*.

Suppose that Ω is a two-sided cell in W_e and y is in W_e . Define $y \leq_{LR} \Omega$ if there is a y' in Ω with $y \leq_{LR} y'$. Then by construction, the span of $\{c'_y \mid y \leq_{LR} \Omega\}$ is a two-sided ideal in \mathcal{H} . We denote this two-sided ideal by $\mathcal{H}_{\overline{\Omega}}$.

The two sided ideals $\mathcal{H}_{\overline{\Omega}}$ define a filtration of \mathcal{H} . In [Lus87, §2], Lusztig has defined a ring J which after extending scalars is isomorphic to \mathcal{H} , but for which the two-sided cells index a decomposition into orthogonal two-sided ideals, rather than a filtration by two-sided ideals.

For x, y , and z in W_e , define $h_{x,y,z}$ in A by $c'_x c'_y = \sum_{z \in W_e} h_{x,y,z} c'_z$. Next, define $a(z)$ to be the least non-negative integer i with the property that $v^i h_{x,y,z}$ is in $\mathbb{Z}[v]$ for all x and y . It is shown in [Lus85, §7] that $a(z) \leq n$. Finally, define $\gamma_{x,y,z}$ to be the constant term of $v^{a(z)} h_{x,y,z}$.

Now let J be the free abelian group with basis $\{j_y \mid y \in W_e\}$ and define a binary operation on J by $j_x * j_y = \sum_{z \in W_e} \gamma_{x,y,z} j_z$. For a two-sided cell Ω in W_e , define J_{Ω} to be the span of $\{j_y \mid y \in \Omega\}$. In [Lus87, §2], Lusztig proved that there are only finitely many two-sided cells in W_e and derived the following properties of $(J, *)$:

- (1) $(J, *)$ is an associative ring with identity.
- (2) J_{Ω} is a two-sided ideal in J and $(J_{\Omega}, *)$ is a ring with identity.
- (3) $J \cong \bigoplus_{\Omega} J_{\Omega}$ (sum over all two-sided cells Ω in W_e).
- (4) There is a homomorphism of A -algebras, $\mathcal{H} \rightarrow J \otimes A$.

Returning to geometry, recall that \mathcal{U} denotes the set of unipotent elements in G and that $\mathcal{B}_u = \{ B' \in \mathcal{B} \mid u \in B' \}$ for u in \mathcal{U} .

Suppose u is in \mathcal{U} , s in G is semisimple, and u and s commute. Let $\langle s \rangle$ denote the smallest closed, diagonalizable subgroup of G containing s and set $\overline{\langle s \rangle} = \langle s \rangle \times \mathbb{C}^*$. In [Lus89b, §2], Lusztig defines an action of $\overline{\langle s \rangle}$ on \mathcal{B}_u using a homomorphism $\mathrm{SL}_2(\mathbb{C}) \rightarrow G$ corresponding to u . Define

$$A_{\mathbb{C}} = A \otimes \mathbb{C}, \quad \mathcal{H}_{\mathbb{C}} = \mathcal{H} \otimes_A A_{\mathbb{C}}, \quad \text{and} \quad \mathcal{K}_{u,s} = \left(K^{\overline{\langle s \rangle}}(\mathcal{B}_u) \otimes \mathbb{C} \right) \otimes_{R(\overline{\langle s \rangle}) \otimes \mathbb{C}} A_{\mathbb{C}}.$$

In [Lus89b, §2], Lusztig defines commuting actions of $\mathcal{H}_{\mathbb{C}}$ and $C(us)$ on $\mathcal{K}_{u,s}$. For an irreducible representation ρ of $C(us)$, let $\mathcal{K}_{u,s,\rho}$ denote the ρ -isotypic component of $\mathcal{K}_{u,s}$, so $\mathcal{K}_{u,s,\rho}$ is an $\mathcal{H}_{\mathbb{C}}$ -module. The next result is proved in [Lus89b, Theorem 4.2].

Theorem 4.6. *Suppose u and s are as above and that ρ is an irreducible representation of $C(us)$ such that $\mathcal{K}_{u,s,\rho} \neq 0$. Then, up to isomorphism, there is a unique simple J -module, E , with the property that when $E \otimes_{\mathbb{C}[v,v^{-1}]} \mathbb{C}(v)$ is considered as an $\mathcal{H}_{\mathbb{C}} \otimes_{\mathbb{C}[v,v^{-1}]} \mathbb{C}(v)$ -module, via the homomorphism $\mathcal{H} \rightarrow J \otimes A$, then $E \otimes_{\mathbb{C}[v,v^{-1}]} \mathbb{C}(v) \cong \mathcal{K}_{u,s,\rho} \otimes_{\mathbb{C}[v,v^{-1}]} \mathbb{C}(v)$.*

Given u , s , and ρ as in the theorem, let $E(u, s, \rho)$ denote the corresponding simple J -module. Since $J \cong \oplus_{\Omega} J_{\Omega}$ and $E(u, s, \rho)$ is simple, there is a unique two-sided cell $\Omega(u, s, \rho)$ with the property that $J_{\Omega(u,s,\rho)} E(u, s, \rho) \neq 0$. The main result in [Lus89b, Theorem 4.8] is the next theorem.

Theorem 4.7. *With the notation as above, the two-sided cell $\Omega(u, s, \rho)$ depends only on the G -conjugacy class of u . Moreover, the rule $(u, s, \rho) \mapsto \Omega(u, s, \rho)$ determines a well-defined bijection between the set of unipotent conjugacy classes in G and the set of two-sided cells in W_e . This bijection has the property that $a(z) = \dim \mathcal{B}_u$ for any z in $\Omega(u, s, \rho)$.*

Using a Springer isomorphism $\mathcal{U} \cong \mathfrak{N}$ we obtain the following corollary.

Corollary 4.8. *There is a bijection between the set of nilpotent G -orbits in \mathfrak{N} and the set of two-sided cells of W_e with the property that if x is in \mathfrak{N} and Ω is the two-sided cell corresponding to the G -orbit $G \cdot x$, then $a(z) = \dim \mathcal{B}_x$ for every z in Ω .*

We can now work out some examples. Let Ω_1 denote the two-sided Kazhdan-Lusztig cell corresponding to the regular nilpotent orbit. Then $a(z) = 0$ for z in Ω_1 and Ω_1 is the unique two-sided Kazhdan-Lusztig cell on which the a -function takes the value 0. Let 1 denote the identity element in W_e . Then it follows immediately from the definitions that $\{1\}$ is a two-sided cell and that $a(1) = 0$. Therefore, $\Omega_1 = \{1\}$.

At the other extreme, let Ω_0 denote the two-sided Kazhdan-Lusztig cell corresponding to the nilpotent orbit $\{0\}$. Then $a(z) = n$ for z in Ω_0 and Ω_0 is the unique two-sided Kazhdan-Lusztig cell on which the a -function takes the value n . Shi [Shi87] has shown that

$$\Omega_0 = \{ y \in W_e \mid a(y) = n \} = \{ y_1 w_0 y_2 \in W_e \mid \ell(y_1 w_0 y_2) = \ell(y_1) + \ell(w_0) + \ell(y_2) \}.$$

The relation \leq_{LR} determines a partial order on the set of two-sided Kazhdan-Lusztig cells and one of the important properties of Lusztig's a function is that $a(y_1) \leq a(y_2)$ whenever $y_2 \leq_{LR} y_1$ (see [Lus85, Theorem 5.4]). Therefore, Ω_1 is the unique maximal two-sided cell and Ω_0 is the unique minimal two-sided cell. It follows that $\mathcal{H}_{\overline{\Omega_1}} = \mathcal{H}$ and that $\mathcal{H}_{\overline{\Omega_0}}$ is the span of $\{ c'_y \mid y \in \Omega_0 \}$.

Summarizing, we have seen that \mathcal{H} is filtered by the two sided ideals $\mathcal{H}_{\overline{\Omega}}$, where Ω runs over the set of two-sided Kazhdan-Lusztig cells in W_e , and that there is a bijection between the set of two-sided cells in W_e and the set of nilpotent orbits in \mathfrak{N} .

Now suppose that \mathfrak{D} is a nilpotent orbit and recall the subvariety $Z_{\overline{\mathfrak{D}}}$ of Z defined in §3.5. Let $i_{\overline{\mathfrak{D}}}: Z_{\overline{\mathfrak{D}}} \rightarrow Z$ denote the inclusion. There are direct image maps, $(i_{\overline{\mathfrak{D}}})_*$ in Borel-Moore homology and in K -theory. It follows from the convolution construction that the images of these maps are two-sided ideals in $H_*(Z)$ and $K^{\overline{G}}(Z)$ respectively. In §3.5 we described the image of $(i_{\overline{\mathfrak{D}}})_*: H_{4n}(Z_{\overline{\mathfrak{D}}}) \rightarrow H_{4n}(Z)$, a two-sided ideal in $H_{4n}(Z)$.

The argument in [KL87, §5] shows that $(i_{\overline{\mathfrak{D}}})_* \otimes id: K^{\overline{G}}(Z_{\overline{\mathfrak{D}}}) \otimes \mathbb{Q} \rightarrow K^{\overline{G}}(Z) \otimes \mathbb{Q}$ is injective. In contrast, $(i_{\overline{\mathfrak{D}}})_*: H_j(Z_{\overline{\mathfrak{D}}}) \rightarrow H_j(Z)$ is an injection when $j = 4n$, but fails to be an injection in general. For example, taking $\overline{\mathfrak{D}} = \mathfrak{D} = \{0\}$, we have that $Z_{\{0\}} = \{0\} \times \mathcal{B} \times \mathcal{B}$ and $\dim H_*(Z_{\{0\}}) = \dim H_*(Z) = |W^2|$. However, $\dim H_{4n}(Z_{\{0\}}) = 1$ and $H_{4n}(Z) = |W|$ and so $(i_{\{0\}})_*: H_j(Z_{\{0\}}) \rightarrow H_j(Z)$ cannot be an injection for all j .

Define $\mathcal{I}_{\overline{\mathfrak{D}}}$ to be the image of $(i_{\overline{\mathfrak{D}}})_*: K^{\overline{G}}(Z_{\overline{\mathfrak{D}}}) \rightarrow K^{\overline{G}}(Z)$, a two-sided ideal in $K^{\overline{G}}(Z)$. There is an intriguing conjectural description of the image of $\mathcal{I}_{\overline{\mathfrak{D}}}$ under the isomorphism $K^{\overline{G}}(Z) \cong \mathcal{H}$ due to Ginzburg [Gin87] that ties together all the themes in this subsection.

Conjecture 4.9. *Suppose that \mathfrak{D} is a G -orbit in \mathfrak{N} and Ω is the two-sided cell in W_e corresponding to \mathfrak{D} as in Corollary 4.8. Then $\phi(\mathcal{I}_{\overline{\mathfrak{D}}}) = \mathcal{H}_{\overline{\Omega}}$, where $\phi: K^{\overline{G}}(Z) \xrightarrow{\cong} \mathcal{H}$ is the isomorphism in Theorem 4.5.*

This conjecture has been proved when G has type A_l by Tanisaki and Xi [TX06]. Xi has recently shown that the conjecture is true after extending scalars to \mathbb{Q} ([Xi08]).

As a first example, consider the case of the regular nilpotent orbit and the corresponding two-sided cell Ω_1 . Then $\overline{\mathfrak{D}} = \mathfrak{N}$, $\mathcal{I}_{\overline{\mathfrak{D}}} = K^{\overline{G}}(Z)$ and $\mathcal{H}_{\overline{\Omega}_1} = \mathcal{H}$. Thus the conjecture is easily seen to be true in this case.

For a more interesting example, consider the case of the zero nilpotent orbit. Then $Z_{\{0\}} = \{0\} \times \mathcal{B} \times \mathcal{B}$. The corresponding two-sided cell, Ω_0 , has been described above and we have seen that $\mathcal{H}_{\overline{\Omega}_0}$ is the span of $\{c'_y \mid y \in \Omega_0\}$.

It is easy to check that $P_{w,w_0} = 1$ for every w in W and thus $c'_{w_0} = v^{-n} \sum_{w \in W} T_w = v^{-n} e$, where e is as in §4.3. Let $\mathcal{H}c'_{w_0}\mathcal{H}$ denote the two sided ideal generated by c'_{w_0} . In [Xi94], Xi has proved the following theorem.

Theorem 4.10. *With the notation as above we have*

$$\phi\left(\mathcal{I}_{\overline{\{0\}}}\right) = \mathcal{H}c'_{w_0}\mathcal{H} = \mathcal{H}_{\overline{\Omega}_0}.$$

4.5. Equivariant K -theory of generalized Steinberg varieties. Suppose \mathcal{P} and \mathcal{Q} are conjugacy classes of parabolic subgroups of G and recall the generalized Steinberg varieties $X^{\mathcal{P},\mathcal{Q}}$ and $X_{0,0}^{\mathcal{P},\mathcal{Q}}$, and the maps $\eta: Z \rightarrow X^{\mathcal{P},\mathcal{Q}}$ and $\eta_1: Z^{\mathcal{P},\mathcal{Q}} = \eta^{-1}(X_{0,0}^{\mathcal{P},\mathcal{Q}}) \rightarrow X_{0,0}^{\mathcal{P},\mathcal{Q}}$ from §2.4. We have a cartesian square of proper morphisms

$$(4.11) \quad \begin{array}{ccc} Z^{\mathcal{P},\mathcal{Q}} & \xrightarrow{k} & Z \\ \eta_1 \downarrow & & \downarrow \eta \\ X_{0,0}^{\mathcal{P},\mathcal{Q}} & \xrightarrow{k_1} & X^{\mathcal{P},\mathcal{Q}} \end{array}$$

where k and k_1 are the inclusions.

The morphism η_1 is smooth and so there is a pullback map in equivariant K -theory, $\eta_1^*: K^{\overline{G}}(X_{0,0}^{\mathcal{P},\mathcal{Q}}) \rightarrow K^{\overline{G}}(Z^{\mathcal{P},\mathcal{Q}})$. We can describe the $R(\overline{G})$ -module structure of $K^{\overline{G}}(Z^{\mathcal{P},\mathcal{Q}})$ and $K^{\overline{G}}(X_{0,0}^{\mathcal{P},\mathcal{Q}})$ using the argument in [Lus98, 7.15] together with a stronger concentration theorem due to Thomason [Tho92, §2].

Theorem 4.12. *The homomorphisms $\eta_1^*: K^{\overline{G}}(X_{0,0}^{\mathcal{P},\mathcal{Q}}) \rightarrow K^{\overline{G}}(Z^{\mathcal{P},\mathcal{Q}})$ and $k_*: K^{\overline{G}}(Z^{\mathcal{P},\mathcal{Q}}) \rightarrow K^{\overline{G}}(Z)$ are injective. Moreover, $K^{\overline{G}}(X_{0,0}^{\mathcal{P},\mathcal{Q}})$ is a free $R(\overline{G})$ -module with rank $|W|^2/|W_P||W_Q|$ and $K^{\overline{G}}(Z^{\mathcal{P},\mathcal{Q}})$ is a free $R(\overline{G})$ -module with rank $|W|^2$.*

The Cellular Fibration Lemma of Chriss and Ginzburg [CG97, 6.2.7] can be used to describe the $R(\overline{G})$ -module structure of $K^{\overline{G}}(X^{\mathcal{P},\mathcal{Q}})$ when $\mathcal{P} = \mathcal{B}$ or $\mathcal{Q} = \mathcal{B}$.

Proposition 4.13. *The equivariant K -group $K^{\overline{G}}(X^{\mathcal{B},\mathcal{Q}})$ is a free $R(\overline{G})$ -module with rank $|W|^2/|W_Q|$.*

We expect that $K^{\overline{G}}(X^{\mathcal{P},\mathcal{Q}})$ is a free $R(\overline{G})$ -module with rank $|W|^2/|W_P||W_Q|$ for arbitrary \mathcal{P} and \mathcal{Q} . We make a more general conjecture about $K^{\overline{G}}(X^{\mathcal{P},\mathcal{Q}})$ after first considering an example in which everything has been explicitly computed.

Consider the very special case when $\mathcal{P} = \mathcal{Q} = \{G\}$. In this case the spaces in (4.11) are well-known:

$$X_{0,0}^{\{G\},\{G\}} \equiv \{0\}, \quad Z^{\{G\},\{G\}} = \overline{Z_{w_0}} = Z_{\{0\}} \cong \mathcal{B} \times \mathcal{B}, \quad \text{and} \quad X^{\{G\},\{G\}} \equiv \mathfrak{N}.$$

Also, $\eta: Z \rightarrow X^{\{G\},\{G\}}$ may be identified with $\mu_z: Z \rightarrow \mathfrak{N}$ and $k: Z^{\{G\},\{G\}} \rightarrow Z$ may be identified with the closed embedding $\mathcal{B} \times \mathcal{B} \rightarrow Z$ by $(B', B'') \mapsto (0, B', B'')$ and so (4.11) becomes

$$\begin{array}{ccc} Z^{\{G\},\{G\}} = Z_{\{0\}} & \xrightarrow{i_{\{0\}}} & Z \\ \downarrow & & \downarrow \mu_z \\ X_{0,0}^{\{G\},\{G\}} = \{0\} & \longrightarrow & \mathfrak{N} \cong X^{\{G\},\{G\}}. \end{array}$$

The image of $(i_{\{0\}})_*: K^{\overline{G}}(Z_{\{0\}}) \rightarrow K^{\overline{G}}(Z)$ is $\mathcal{I}_{\overline{\{0\}}}$ and we saw in Theorem 4.10 that $\mathcal{I}_{\overline{\{0\}}} \cong \mathcal{H}c'_{w_0} \mathcal{H} = \mathcal{H}_{\overline{\Omega_0}}$.

Ostrik [Ost00] has described the map $(\mu_z)_*: K^{\overline{G}}(Z) \rightarrow K^{\overline{G}}(X^{\{G\},\{G\}})$. Recall that $W_e = X(T) \rtimes W$. Because the fundamental Weyl chamber is a fundamental domain for the action of W on $X(T) \otimes \mathbb{R}$, it follows that each (W, W) -double coset in W_e contains a unique element in X^+ . Also, each (W, W) -double coset in W_e contains a unique element with minimal length. For λ in X^+ we let m_λ denote the element with minimal length in the double coset $W\lambda W$.

Theorem 4.14. *For x in W_e , $(\mu_z)_*(c'_x) = 0$ unless $x = m_\lambda$ for some λ in X^+ . Moreover, the map $(\mu_z)_*: K^{\overline{G}}(Z) \rightarrow K^{\overline{G}}(X^{\{G\},\{G\}})$ is surjective and $\{(\mu_z)_*(c'_{m_\lambda}) \mid \lambda \in X^+\}$ is an A -basis of $K^{\overline{G}}(X^{\{G\},\{G\}})$.*

Notice that the theorem is the K -theoretic analog of Theorem 3.16 in the very special case we are considering.

To prove this result, Ostrik uses the description of Z as a fibred product and the two corresponding factorizations of μ_z :

$$(4.15) \quad \begin{array}{ccc} Z = \tilde{\mathfrak{N}} \times_{\mathfrak{N}} \tilde{\mathfrak{N}} & \longrightarrow & X^{\mathcal{B}, \{G\}} \cong \tilde{\mathfrak{N}} \\ \downarrow & & \downarrow \\ \tilde{\mathfrak{N}} \cong X^{\{G\}, \mathcal{B}} & \longrightarrow & X^{\{G\}, \{G\}} \cong \mathfrak{N}. \end{array}$$

It follows from the construction of the isomorphism $K^{\overline{G}}(Z) \cong \mathcal{H}$ given by Chriss and Ginzburg [CG97, §7.6] (see the end of §4.3) that after applying the functor $K^{\overline{G}}$ to (4.15) the resulting commutative diagram of equivariant K -groups may be identified with the following commutative diagram subspaces of \mathcal{H} :

$$(4.16) \quad \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H}c'_{w_0} \\ \downarrow & & \downarrow \\ c'_{w_0}\mathcal{H} & \longrightarrow & c'_{w_0}\mathcal{H}c'_{w_0} \end{array}$$

where the maps are given by the appropriate right or left multiplication by c'_{w_0} .

We conclude with a conjecture describing $K^{\overline{G}}(X^{\mathcal{P}, \mathcal{Q}})$ for arbitrary \mathcal{P} and \mathcal{Q} . Recall from §3.7 that $X^{\mathcal{P}, \mathcal{Q}} \cong \tilde{\mathfrak{N}}_{\mathcal{P}} \times_{\mathfrak{N}} \tilde{\mathfrak{N}}^{\mathcal{Q}}$. The projection $\mu: \tilde{\mathfrak{N}} \rightarrow \mathfrak{N}$ factors as $\tilde{\mathfrak{N}} \xrightarrow{\eta^{\mathcal{P}}} \tilde{\mathfrak{N}}^{\mathcal{P}} \xrightarrow{\xi^{\mathcal{P}}} \mathfrak{N}$ where $\eta^{\mathcal{P}}(x, gBg^{-1}) = (x, gPg^{-1})$ and $\xi^{\mathcal{P}}(x, gPg^{-1}) = x$. Using this factorization, we may expand diagram (4.15) to a 3×3 diagram with $X^{\mathcal{P}, \mathcal{Q}}$ in the center:

$$(4.17) \quad \begin{array}{ccccc} Z & \longrightarrow & X^{\mathcal{B}, \mathcal{Q}} & \longrightarrow & \tilde{\mathfrak{N}} \\ \downarrow & & \downarrow & & \downarrow \\ X^{\mathcal{P}, \mathcal{B}} & \longrightarrow & X^{\mathcal{P}, \mathcal{Q}} & \longrightarrow & \tilde{\mathfrak{N}}^{\mathcal{P}} \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathfrak{N}} & \longrightarrow & \tilde{\mathfrak{N}}^{\mathcal{Q}} & \longrightarrow & \mathfrak{N}. \end{array}$$

Let w_P and w_Q denote the longest elements in W_P and W_Q respectively. Comparing (4.15), (4.16), and (4.17), we make the following conjecture. This conjecture is a K -theoretic analog of (3.17) and Conjecture 3.19.

Conjecture 4.18. *With the notation above, $K^{\overline{G}}(X^{\mathcal{P}, \mathcal{Q}}) \cong c'_{w_P} \mathcal{H} c'_{w_Q}$.*

If the conjecture is true, then after applying the functor $K^{\overline{G}}$ to (4.17) the resulting commutative diagram of equivariant K -groups may be identified with the following commutative

diagram of subspaces of \mathcal{H} :

$$(4.19) \quad \begin{array}{ccccc} \mathcal{H} & \longrightarrow & \mathcal{H}c'_{w_Q} & \longrightarrow & \mathcal{H}c'_{w_0} \\ \downarrow & & \downarrow & & \downarrow \\ c'_{w_P}\mathcal{H} & \longrightarrow & c'_{w_P}\mathcal{H}c'_{w_Q} & \longrightarrow & c'_{w_P}\mathcal{H}c'_{w_0} \\ \downarrow & & \downarrow & & \downarrow \\ c'_{w_0}\mathcal{H} & \longrightarrow & c'_{w_0}\mathcal{H}c'_{w_Q} & \longrightarrow & c'_{w_0}\mathcal{H}c'_{w_0}. \end{array}$$

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